Linear Inviscid Damping for Monotone Shear Flows, Boundary Effects and Sharp Sobolev Regularity

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Introduction

The topic of this thesis is the analysis of the linear stability and long-time asymptotic behavior of solutions to the 2D incompressible Euler equations

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= 0, \\ \nabla \times v &= \omega, \\ \nabla \cdot v &= 0, \end{aligned}$$

which describe the dynamics of an inviscid incompressible fluid. More specifically, the focus is on solutions close to monotone shear flows and the phenomenon of *linear inviscid damping*. Here, we consider both an infinite and finite periodic channel, the latter with impermeable walls.

The Euler equations possess many conserved quantities, among them the kinetic energy, the enstrophy and entropy, and in particular exhibit neither dissipation nor entropy increase. As shown by Arnold, [**Arn66b**], they even possess the structure of an infinite-dimensional Hamiltonian system on the Lie algebra of smooth volume-preserving diffeomorphisms. It was thus a very surprising observation of Kelvin, [**Kel87**], and later Orr, [**Orr07**], that small perturbations to Couette flow, i.e. the linear shear $v(t,x,y) = (y,0)$, are damped back to a (possibly different) shear flow. This phenomenon is commonly called *inviscid damping* in analogy to *Landau damping* in plasma physics.

If one considers the linearization around Couette flow in an infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, the equations can be solved explicitly using the method of characteristics and Fourier methods. One can thus directly compute that a perturbation $(v,\omega) \in L^2 \times H^2$ is damped to a shear flow with algebraic rates:

$$\begin{aligned} \|v_1 - \langle v_1 \rangle_x\|_{L^2} \leq & \mathcal{O}(t^{-1}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H^{-1}_x H^1_y}, \\ \|v_2\|_{L^2} \leq & \mathcal{O}(t^{-2}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H^{-1}_x H^2_y}, \end{aligned}$$

and that these decay rates are optimal. Going beyond the explicitly solvable (and in this sense trivial) setting of linearized Couette flow, however has remained mostly open until recently.

- In [**BM10**], Bouchet and Morita give heuristic results suggesting that linear damping and stability results should also hold for general monotone shear flows. However, their methods are highly non-rigorous and lack necessary regularity, stability and error estimates, as discussed in [**Zil12**]. In particular, even supposing their asymptotic computations were valid, they do not yield the above decay rates.
- Lin and Zeng, [**LZ11**], use the explicit solution of linearized Couette flow to establish linear damping also in a finite periodic channel. Furthermore, they show the existence of non-trivial stationary solutions to the full 2D Euler equations in arbitrarily small H^s neighborhoods of Couette flow for any $s < \frac{3}{2}$. As a consequence, nonlinear inviscid damping can not hold in such low regularity.

- Recently, following the work of Villani and Mouhot, [**MV11**], on nonlinear Landau damping, Masmoudi and Bedrossian, [**BM13b**], have proven nonlinear inviscid damping for small Gevrey (see Definition 3.1) perturbations to Couette flow in an infinite periodic channel. We briefly discuss their results and the additional challenges in the nonlinear setting in Chapter 6.

As the main result of this thesis, in Chapters 4 and 5, we, for the first time, rigorously establish linear inviscid damping for a general class of monotone shear flows. Here, we treat both the setting of an infinite periodic channel with period L, $\mathbb{T}_L \times \mathbb{R}$, and a finite periodic channel, $\mathbb{T}_L \times [0,1]$, with impermeable walls. In the latter setting, we show that boundary effects play a non-negligible role and prove (almost) sharp results on the stability in fractional Sobolev spaces.

As a first result in this direction, in the author's Master's thesis, [**Zil12**], it has been shown that damping results of the above form also hold for general, regular, strictly monotone shear flows, *assuming* one can control the regularity of the vorticity moving with the underlying shear flow.

THEOREM 0.1 (Damping). *Let $U(y)$ be a strictly monotone, regular shear flow, i.e. $U' > c > 0$ and $U' \in W^{2,\infty}$. Then for any solution ω of the linearized 2D Euler equations in either the infinite periodic channel or the finite periodic channel, denoting*

$$W(t,x,y) := \omega(t, x - tU(y), y) - \langle \omega_0 \rangle_x (y),$$

the perturbation to the velocity field is controlled by

$$\begin{aligned}\|v_1(t) - \langle v_1 \rangle_x\|_{L^2} \leq& \mathcal{O}(t^{-1}) \|W(t)\|_{H^{-1}_x H^1_y},\\ \|v_2(t)\|_{L^2} \leq& \mathcal{O}(t^{-2}) \|W(t)\|_{H^{-1}_x H^2_y}.\end{aligned}$$

Assuming control of $\|W(t)\|_{L^2_x H^2_y}$ uniformly in time, the velocity perturbation hence decays with the optimal algebraic rates. As a consequence, under the same *assumption*, it can be shown that ω converges to a free solution of the underlying transport equation, i.e. that W converges to some asymptotic profile.

THEOREM 0.2 (Scattering). *Let W be a solution of the linearized 2D Euler equations in either the infinite periodic channel or finite periodic channel and suppose that $U'' \in L^\infty$ and that*

$$\|v_2(t)\|_{L^2} = \mathcal{O}(t^{-1-\epsilon}),$$

for some $\epsilon > 0$. Then there exists $W_\infty \in L^2$, such that

$$W(t) \xrightarrow{L^2} W_\infty,$$

as $t \to \infty$.

In Corollary 4.2 in Section 4 of Chapter 4, this result is further improved to arbitrary L^2 initial data:

THEOREM 0.3 (L^2 scattering). *Let $\Omega = \mathbb{T}_L \times \mathbb{R}$ or $\mathbb{T}_L \times [0,1]$ and suppose that there exists c such that*

$$0 < c < U' < c^{-1} < \infty,$$

and further suppose that

$$\|U''(U^{-1}(\cdot))\|_{W^{2,\infty}} L$$

is sufficiently small. Then, for every $\omega_0 \in L^2(\Omega)$, there exists $W^\infty \in L^2(\Omega)$, such that the solution, ω, of the linearized Euler equations on Ω with initial datum ω_0 satisfies

$$W(t,x,y) := \omega(t, x - tU(y), y) \xrightarrow{L^2_{xy}} W^\infty,$$

as $t \to \infty$.

More precise statements, proofs and further similar results are given in Section 1 of Chapter 4. It is thus shown that linear inviscid damping, like Landau damping, fundamentally is a problem of stability and regularity. We stress that the uniform damping results necessarily cost regularity and that stability results hence rely on a detailed analysis of fine properties of the dynamics.

As the first main result of this thesis we establish stability of the linearized Euler equations around regular, strictly monotone shear flows in an infinite periodic channel for arbitrarily high Sobolev spaces. For this purpose, we first introduce a model problem to analyze the damping mechanism. Subsequently, we introduce a decaying Fourier weight adapted to the dynamics. As a consequence of the stability result, we establish linear inviscid damping with optimal rates and scattering for a large class of monotone shear flows.

THEOREM 0.4 (Stability for $\mathbb{T}_L \times \mathbb{R}$). *Let $j \in \mathbb{N}$ and suppose that*

$$0 < c < U' < c^{-1} < \infty,$$

and $U''(U^{-1}(\cdot)) \in W^{j+1,\infty}(\mathbb{R})$. Suppose further that

$$L\|U''(U^{-1}(\cdot))\|_{W^{j+1,\infty}}$$

is sufficiently small. Then, for any $m \in \mathbb{N}$ and any $\omega_0 \in H^j_y H^m_x$,

$$\|W(t)\|_{H^j_y H^m_x} \lesssim \|\omega_0\|_{H^j_y H^m_x},$$

uniformly in time.

When considering a finite channel instead, we show that such a regularity result can not hold in arbitrary Sobolev spaces, but rather that in general boundary derivatives of W will develop (logarithmic) singularities as $t \to \infty$ (c.f. Propositions 5.1 and 5.5). The regularity is thus limited to fractional Sobolev spaces, which is shown to be sharp (c.f. Theorems 4.14 and 5.1). More precisely, instability is proven for the standard fractional Sobolev spaces, $H^s_y([0,1])$. For the stability results, for technical reasons, we instead consider the periodic fractional Sobolev spaces $H^s_y(\mathbb{T})$ and additionally require the coefficient functions, U', U'', corresponding to the shear flow to have regular periodic extensions. As discussed in Remark 9 in Section 1 in Chapter 5, these periodicity assumptions can probably be relaxed.

THEOREM 0.5 (Sharp stability in $\mathbb{T}_L \times (0,1)$). *Let $U', U'' \in W^{3,\infty}([0,1])$ and suppose that there exists $c > 0$ such that*

$$0 < c < |U'| < c^{-1} < \infty,$$

and that

$$L\|U''\|_{W^{3,\infty}}$$

is sufficiently small. Then, for any $m \in \mathbb{N}$ and any $\omega_0 \in H^m_x H^1_y(\mathbb{T}_L \times [0,1])$

$$\|W(t)\|_{H^m_y H^1_y} \lesssim \|\omega_0\|_{H^m_x H^1_y},$$

uniformly in time.

Suppose further that $U', U'' \in W^{3,\infty}(\mathbb{T})$, i.e. that the periodic extensions are also regular, and let

$$L\|U''\|_{W^{3,\infty}(\mathbb{T})}$$

be sufficiently small. Then also for any $s < \frac{3}{2}, m \in \mathbb{N}$ and any $\omega_0 \in H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T})$,

$$\|W(t)\|_{H^m_x H^s_y} \lesssim \|\omega_0\|_{H^m_x H^s_y},$$

uniformly in time.

If $\omega_0|_{y=0,1}$ and $U''|_{y=0,1}$ are non-trivial, then for any $s > \frac{3}{2}$ and any $m \in \mathbb{N}$,

$$\sup_t \|W(t)\|_{H^m_x H^s_y(\mathbb{T}_L \times [0,1])} = \infty.$$

Restricting to perturbations with $\omega_0|_{y=0,1} = 0$, the stability and instability results can be improved by one derivative, which is shown to be a sharp restriction for general perturbations (c.f. Theorems 4.15 and 5.4).

Theorem 0.6 (Sharp stability in $\mathbb{T}_L \times (0,1)$ under restricted perturbations). *Let $U', U'' \in W^{4,\infty}([0,1])$ and suppose that there exists $c > 0$ such that*

$$0 < c < |U'| < c^{-1} < \infty,$$

and that

$$L\|U''\|_{W^{4,\infty}}$$

is sufficiently small. Then, for any $m \in \mathbb{N}$ and any $\omega_0 \in H^m_x H^2_y(\mathbb{T}_L \times [0,1])$ with $\omega_0|_{y=0,1} = 0$,

$$\|W(t)\|_{H^m_y H^2_y} \lesssim \|\omega_0\|_{H^m_x H^2_y},$$

uniformly in time.

Suppose further that $U', U'' \in W^{4,\infty}(\mathbb{T})$, i.e. that the periodic extensions are also regular, and let

$$L\|U''\|_{W^{4,\infty}(\mathbb{T})}$$

be sufficiently small. Then also for any $s < \frac{5}{2}, m \in \mathbb{N}$ and any $\omega_0 \in H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T})$ with $\omega_0|_{y=0,1} = 0$,

$$\|W(t)\|_{H^m_x H^s_y} \lesssim \|\omega_0\|_{H^m_x H^s_y},$$

uniformly in time. Furthermore, $\lim_{t\to\infty} \partial_y W|_{y=0,1}$ exists.

Suppose that the limit $\lim_{t\to\infty} U''\partial_y W|_{y=0,1}$ exists and is non-trivial. Then, for any $s > \frac{5}{2}$ and any $m \in \mathbb{N}$,

$$\sup_t \|W(t)\|_{H^m_x H^s_y(\mathbb{T}_L \times [0,1])} = \infty.$$

As a consequence, we obtain linear inviscid damping in a finite periodic channel for a large class of monotone shear flows. Subsequently, we discuss the implications of the singularity formation at the boundary for the nonlinear problem, where very high regularity is used to control nonlinear interactions.

We conclude this introduction with a short overview of the organization of the thesis's chapters:

- In Chapter 1, we introduce the Euler equations and their various formulations and briefly review classical stability results, the Hamiltonian structure and conserved quantities. Furthermore, we introduce the Vlasov-Poisson equations and discuss their structural commonalities with Euler's equations.
- In Chapter 2, free transport serves to introduce the phase-mixing mechanism underlying Landau damping and inviscid damping. It also provides a heuristic model to discuss the expectations for damping rates and the role of regularity.
- In Chapter 3, we briefly sketch how to prove linear Landau damping and discuss the additional challenges arising for the nonlinear dynamics, following [**MV11**], [**BMM13**].
- In Chapter 4, we establish linear inviscid damping for general monotone shear flows:
  - As a first step, following [**Zil12**], we show that, like Landau damping, (linear) inviscid damping is fundamentally a problem of regularity and stability. More precisely, *assuming* control of W in Sobolev regularity, decay estimates are proven for a large class of general, regular, strictly monotone shear flows. We stress that, as uniform damping estimates necessarily lose regularity, stability results need to make use of finer properties of the damping mechanism.
  - In the setting of an infinite channel, $\mathbb{T}_L \times \mathbb{R}$, we first introduce a model problem to study finer properties of the dynamics. Subsequently, we construct a Fourier weight, which is adapted to the dynamics, and prove stability in arbitrarily high Sobolev norms, $H^m_x H^j_y$.
  - The setting of a finite channel, $\mathbb{T}_L \times [0,1]$, is shown to be not only technically more challenging, but to exhibit qualitatively different behavior due to boundary effects. Here, we prove stability in $H^m_x H^1_y$ for generic perturbations. In contrast to the setting of an infinite channel, where stability holds in arbitrarily high Sobolev spaces, we show that in a finite channel $H^m_x H^2_y$ stability necessarily needs to restrict to perturbations, ω_0, with zero Dirichlet data, $\omega_0|_{y=0,1} = 0$. For such perturbations, hence linear inviscid damping with the optimal rates and scattering hold.

  An earlier version of this chapter has been made available as a preprint on arXiv, [**Zil14**].
- In Chapter 5, we further study boundary effects and the formation of singularities. There, we show that the stability results under general perturbations can be improved to the periodic fractional Sobolev spaces $H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T}), s < \frac{3}{2}$. Furthermore, $s = \frac{3}{2}$ is shown to be critical in the sense that for any $s > \frac{3}{2}$, due boundary effects, the $H^m_x H^s_y(\mathbb{T}_L \times [0,1])$ norm of W blows up as $t \to \infty$.

  Restricting to perturbations such that $\omega_0|_{y=0,1}$, the critical space in y is shown to be $H^{\frac{5}{2}}_y$. That is, we prove stability for $s < \frac{5}{2}$ and prove that, in general, for any $s > \frac{5}{2}$, due boundary effects, the $H^m_x H^s_y$ norm of W blows up as $t \to \infty$,
- Finally, in Chapter 6, we discuss consistency and implications for the nonlinear dynamics. There, we also briefly review the existing literature on nonlinear inviscid damping for Couette flow and discuss the additional challenges arising for the nonlinear dynamics.

CHAPTER 1

The Euler and Vlasov-Poisson equations

In this chapter, we introduce the various formulations of Euler's equations and briefly review some of the classical results on stability, well-posedness and long-time asymptotic behavior of solutions. Subsequently, we introduce the Vlasov-Poisson equations of plasma physics and discuss the structural similarities of both equations.

1. The incompressible Euler equations

The incompressible Euler equations model the evolution of an incompressible, inviscid fluid. While physical fluids are neither perfectly incompressible nor perfectly inviscid, for many applications Euler's equations nevertheless provide a good description. In this section, we introduce the incompressible 2D Euler equations in their various formulations and discuss their structure and conserved quantities. We also briefly discuss the 3D Euler equations and comment on some of the additional challenges. For simplicity, in this chapter we phrase all results for the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively and only briefly comment on the modifications for regular bounded domains, such as imposing boundary conditions on the velocity field. Additionally, as can be seen by the results of Buckmaster, De Lellis and Székelyhidi, [**BDLSJ14**], low regularity solutions can behave in unexpected and somewhat unphysical ways, such as not preserving the kinetic energy. In the following, we therefore argue formally and assume that all functions are "smooth and rapidly decaying at infinity". For a more precise and extensive discussion we refer to the book of Majda and Bertozzi, [**MB01**].

Subsequently, we review the stability results by Arnold, Fjørtoft and Rayleigh. In particular, we recall that the Euler equations have an (infinite-dimensional) Hamiltonian structure and possess many conserved quantities.

1.1. The velocity formulation. Let $n = 2, 3$, then the incompressible Euler equations on $\mathbb{R}^n$ in velocity formulation are given by

$$\begin{aligned} \partial_t v + v \cdot \nabla v &= \nabla p, \\ \nabla \cdot v &= 0, \end{aligned} \tag{Euler}$$

where the velocity

$$v : \mathbb{R}^n \times (0, T) \to \mathbb{R}^n,$$

is a smooth vector field and the pressure

$$p : \mathbb{R}^n \times (0, T) \to \mathbb{R},$$

is a smooth scalar function.

The derivative

$$D_t := (\partial_t + v \cdot \nabla),$$

is called a convective derivative, i.e. the derivative along particle trajectories, as we discuss in more detail in Section 1.4. The gradient of the pressure p

$$F = \nabla p$$

is a force field, which acts as a Lagrange multiplier to ensure the incompressibility constraint

$$\nabla \cdot v = 0.$$

We expand on this characterization in Section 2.1. The Euler equations are thus structurally similar to Newton's second law

$$m\frac{d}{dt}v = F,$$

and, indeed, can be formally derived from it by studying the motion of small domains of the fluid. A more rigorous derivation of Euler's equations such as the hydrodynamic limit of Boltzmann's equations, [**SR09**], is, however, technically much more challenging.

This thesis's focus is on the stability and asymptotic behavior of solutions to the two-dimensional Euler equations close to shear flow solutions $v = (U(y), 0)$, which we discuss in Section 1.5. In this chapter, we work in slightly more generality and consider solutions in two or three dimensions. The Euler equations in velocity formulation thus involve either 3 or 4 unknowns, respectively. In order to reduce the number of unknowns, we may express p in terms of v. Taking the divergence of the first equation and using that $\nabla \cdot v = 0$, p satisfies

$$\Delta p = \nabla \cdot (v \cdot \nabla v).$$

Imposing suitable boundary conditions on the Laplacian, p and thus Euler's equations can be expressed in terms of v only. However, the dependence on v is then even more nonlinear and also non-local.

In the following, we consider another common reduction to eliminate p, which is called the vorticity-stream formulation.

1.2. The vorticity-stream formulation. In this section, we introduce the vorticity-stream formulation of Euler's equations, which focuses on the evolution of the *vorticity* $\omega = \nabla \times v$. In three dimensions the vorticity is a vector field taking values in $\mathbb{R}^3$, while in two dimensions ω is a scalar function, which greatly simplifies the equations. In the following, we therefore first discuss the three dimensional case and subsequently study the simplifications for two dimensions.

Let $n = 3$, then the vorticity

$$\omega = \nabla \times v : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3$$

satisfies

$$\partial_t \omega + \nabla \times (v \cdot \nabla v) = 0,$$

where we used that

$$\nabla \times (\nabla p) = 0.$$

We compute

$$\nabla \times (v \cdot \nabla v) = v \cdot \nabla \omega + \omega \cdot \nabla v.$$

The first term is of transport type, i.e. ω is transported by the velocity field v. The second term is called the *vortex-stretching term*, which is of considerable interest for well-posedness theory and blow-up in three dimensions (c.f. [**BKM84**] and [**MB01**, Chapter 2]).

Thus, the functions v and ω satisfy

$$\begin{aligned}\partial_t \omega + v \cdot \nabla \omega + \omega \cdot \nabla v &= 0,\\ \nabla \cdot v &= 0,\\ \nabla \times v &= \omega,\end{aligned}$$

which involves 6 unknowns (v, ω) at the moment, but is linear in both v and ω. In order to express the equations in terms of ω only we have to solve

$$\begin{aligned}\nabla \cdot v &= 0,\\ \nabla \times v &= \omega,\end{aligned} \tag{1}$$

for v.

PROPOSITION 1.1 (Hodge decomposition, [**MB01**, section 2.4]). *Let $\omega \in L^2(\mathbb{R}^3; \mathbb{R}^3)$ be a smooth vector field, vanishing sufficiently rapidly at infinity.*

- *The equations* (1) *have a smooth solution v, vanishing rapidly at infinity, if and only if*
$$\nabla \cdot \omega = 0.$$
- *If $\nabla \cdot \omega = 0$, then v is determined constructively by*
$$v = -\nabla \times \psi,$$
where ψ solves
$$\Delta \psi = \omega,$$
and there is an explicit kernel $K_3(x)$
$$K_3(x)h = \frac{1}{4\pi} \frac{x \times h}{|x|^3}, \quad h \in \mathbb{R}^3,$$
such that
$$v(x) = \int_{\mathbb{R}^3} K_3(x - y)\omega(y)dy.$$

PROPOSITION 1.2 (Vorticity-stream formulation in $\mathbb{R}^3$, [**MB01**, section 2.4]). *For smooth flows that vanish sufficiently rapidly at infinity, the system of equations*

$$\begin{aligned}\partial_t \omega + v \cdot \nabla \omega - \omega \cdot \nabla v &= 0,\\ \omega|_{t=0} = \omega_0 &= \nabla \times v_0,\end{aligned}$$

with v determined by the previous proposition, is equivalent to the Euler equations.

Compared to the Euler equations in velocity formulation, this system of equations contains only three unknowns ω and the dependence of v on ω is linear. As we discuss in the following, in two dimensions the vorticity stream formulation yields an even greater simplification.

Let $n = 2$, then the vorticity

$$\omega = \nabla \times v : \mathbb{R}^2 \times (0, T) \to \mathbb{R}$$

satisfies

$$\partial_t \omega + v \cdot \nabla \omega = 0,$$

where we used that

$$\nabla \times (\nabla p) = 0$$

and

$$\nabla \times (v \cdot \nabla v) = v \cdot \nabla(\nabla \times v).$$

The vorticity is thus only transported by v and there is no vortex-stretching term. We discuss the implications of this in Section 2.2.

In order to express v in terms of ω, we note that

$$0 = \nabla \cdot v = \nabla \times (-v_2, v_1) =: \nabla \times v^{\perp}.$$

There thus exists a potential ϕ, called the *stream function*, such that

$$v^{\perp} = \nabla \phi \Leftrightarrow v = \nabla^{\perp} \phi.$$

Taking the curl of this equation we thus obtain

$$\begin{aligned} \Delta \phi &= \omega, \\ \nabla^{\perp} \phi &= v, \end{aligned}$$

where, in the case of a bounded domain, ϕ is chosen to satisfy given boundary conditions for v.

Definition 1.1 (Vorticity-stream formulation). The 2D Euler equations in *vorticity-stream formulation* are given by

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= 0, \\ \Delta \phi &= \omega, \\ v &= \nabla^{\perp} \phi, \end{aligned} \tag{E}$$

where, in the case of a bounded domain, the boundary conditions for ϕ or v respectively are further specified.

Proposition 1.3 (Vorticity-stream formulation for $\mathbb{R}^2$ [**MB01**, page 45]). *For smooth 2D flows vanishing sufficiently rapidly at infinity, the velocity formulation is equivalent to the vorticity-stream formulation.*

The preceding two-dimensional results can be adapted to settings involving boundaries.

In this thesis we are primarily interested in the following two settings:

- The infinite periodic channel, $\mathbb{T} \times \mathbb{R}$.
- The finite periodic channel, $\mathbb{T} \times [0,1]$. Here the natural boundary conditions are given by impermeable walls, i.e. $v_2(x,y) = 0$ whenever $y \in \{0,1\}$.

As we discuss in Chapters 4 and 5, in the latter setting the boundary conditions strongly influence the dynamics.

In the following section, we consider symmetries of the Euler equations in two and three dimensions.

1.3. Symmetries and Galilean invariance. Like many physically relevant models, the Euler equations enjoy many symmetries, including invariance under Galilean transformations.

Lemma 1.1 ([**MB01**, page 3]). *Let (v,p) be a solution to Euler's equations on $\mathbb{R}^n$, $n = 2$ or 3. Then the following transformations also yield solutions:*

(1) Galilean invariance: For any $c \in \mathbb{R}^n$, the pair (v_c, p_c) with

$$\begin{aligned} v_c(t,x) &= v(t, x - tc) + c, \\ p_c(t,x) &= p(t, x - tc), \end{aligned}$$

is also a solution.

(2) Rotation symmetry: Let $Q \in SO(n)$, then the pair (v_Q, p_Q) with

$$\begin{aligned} v_Q(t,x) &= Q^T v(t, Qx), \\ p_Q(t,x) &= p(t, Qx), \end{aligned}$$

is also a solution.

(3) Scale invariance: Let $\lambda, \tau \in \mathbb{R} \setminus \{0\}$, then the pair $(v_{\lambda,\tau}, p_{\lambda,\tau})$ with

$$v_{\lambda,\tau}(t,x) = \frac{\lambda}{\tau} v(\frac{x}{\lambda}, \frac{t}{\tau}),$$
$$p_{\lambda,\tau}(t,x) = \frac{\lambda^2}{\tau^2} p(\frac{x}{\lambda}, \frac{t}{\tau}),$$

is also a solution.

We remark that, in the case of a domain $\Omega \subset \mathbb{R}^2$ different from $\mathbb{R}^2$ or the (in)finite periodic channel, not all symmetries leave the underlying domain invariant, but rather yield solutions on a modified space.

In particular, in this way it is possible to rescale any given infinite channel, $\mathbb{T}_L \times \mathbb{R}$, to the one with a prescribed (unit) period, $\mathbb{T} \times \mathbb{R}$. In the case of a finite channel, we instead normalize the width, i.e. $\mathbb{T}_L \times [0,b]$ is rescaled to $\mathbb{T}_{Lb} \times [0,1]$.

1.4. The Lagrangian and Eulerian perspective. In both the velocity and vorticity-stream formulation, we considered the evolution of the fluid with respect to a fixed, given spatial coordinate system. In this section, we adopt a different perspective and consider particle markers α "moving with the flow" as our coordinates.

DEFINITION 1.2 (Flow map). Let $n = 2, 3$ and let

$$v : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n$$

be a given, smooth velocity field. Then the flow-map $X : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n$ is defined as the unique solution of

$$\partial_t X(\alpha,t) = v(X(\alpha,t),t),$$
$$X(\alpha,0) = \alpha.$$

Here, α is the initial position of a particle.

The spatially fixed perspective is called *Eulerian*, while the one moving with the flow is called *Lagrangian*. As we discuss in the following, for sufficiently smooth velocity fields it is equivalent to consider the evolution of X and of the associated velocity field. In this way we introduce a *particle-trajectory formulation* of Euler's equations. Here, again the two-dimensional setting is greatly simplified compared to the three-dimensional setting.

Before further studying this formulation, we briefly elaborate on our previous remark that convective derivative

$$D_t = \partial_t + v \cdot \nabla$$

is a derivative "along the flow of v".

LEMMA 1.2. *Let $n = 2, 3$ and let*

$$f : \mathbb{R}^n \times (0,T) \mapsto \mathbb{R}$$

be any smooth function, v a smooth velocity field and X its associated flow map. Then

$$D_t f(t,\alpha) = \frac{d}{dt}|_{t=0} f(t, X(t,\alpha)).$$

PROOF. This follows immediately by the chain rule and the initial condition

$$X(\alpha,0) = \alpha.$$

□

The incompressibility condition of Euler's equations

$$\nabla \cdot v = 0,$$

further implies that X is volume-preserving.

LEMMA 1.3 ([**MB01**, page 5]). *Let X be the flow corresponding to a smooth velocity field v. Then,*

$$\frac{d}{dt} \det(\nabla_\alpha X) = (\nabla_x \cdot v)|_{(X(\alpha,t),t)} \det(\nabla_\alpha X).$$

In particular, if $\nabla \cdot v \equiv 0$, then $\det(\nabla_\alpha X) \equiv 1$.

In oder to make use of this *particle-trajectory formulation* of Euler's equations, we need to be able to express $v(t, X(\alpha, t))$ in terms of X and the initial data only (c.f. [**MB01**, section 2.5]). In two dimensions the vorticity is transported by the velocity field:

$$\partial_t \omega + v \cdot \nabla \omega = 0,$$

and thus ω satisfies

$$\omega(t, X(t, \alpha)) = \omega_0(\alpha).$$

In particular, given X and ω_0, we can compute ω and thus v. Here the kernel K_2 of the mapping

$$\omega \mapsto v = \int_{\mathbb{R}^2} K_2(x-y)\omega(t,y)dy$$

is given by (c.f. [**MB01**, page 81])

$$K_2(x) = \frac{1}{2\pi}(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2}), \quad x \in \mathbb{R}^2.$$

Using the fact that X is volume-preserving, we thus compute

$$\begin{aligned} v(x,t) &= \int K_2(x-y)\omega(t,y)dy = \int K_2(x - X(t,\alpha'))\omega(t, X(t,\alpha'))d\alpha' \\ &= \int K_2(x - X(\alpha',t))\omega_0(\alpha')d\alpha'. \end{aligned}$$

In two dimensions the flow map X thus is given by the solution of the integro-differential equation

$$\begin{aligned} \frac{d}{dt}X(\alpha,t) &= \int K_2(X(\alpha,t) - X(\alpha',t))\omega_0(\alpha')d\alpha', \\ X(\alpha,0) &= \alpha. \end{aligned}$$

In three dimensions, as discussed in Section 1.2 the vorticity is not only transported but can be stretched by the flow. It can be shown (c.f [**MB01**, page 82]) that in this case ω satisfies

$$\omega(X(\alpha,t),t) = \nabla_\alpha X(\alpha,t)\omega_0(\alpha).$$

Proceeding as previously, with K_3 as in Section 1.2, we obtain that X satisfies

$$\begin{aligned} \frac{d}{dt}X(\alpha,t) &= \int K_3(X(\alpha,t) - X(\alpha',t))\nabla_\alpha X(\alpha',t)\omega_0(\alpha')d\alpha', \\ X(\alpha,0) &= \alpha. \end{aligned}$$

The equivalence of this particle-trajectory formulation and the velocity formulation of the 3D Euler's equations is given by the following proposition.

Proposition 1.4 ([**MB01**, page 83]). *Let v_0 be a smooth 3D velocity field with $\nabla \cdot v_0 = 0$ and $\omega_0 = \nabla \times v_0$. Let X be a solution of the particle-trajectory formulation with*

$$\frac{d}{dt}|_{t=0} X = v_0,$$

and define

$$v(t,x) = \int_{\mathbb{R}^3} K_3(x - X(\alpha,t)) \nabla_\alpha X(\alpha', t)\omega_0(\alpha')d\alpha'.$$

Then v is a solution of Euler's equations with initial datum v_0. Thus the particle-trajectory formulation is equivalent to the velocity formulation for sufficiently smooth solutions with rapidly decaying vorticity ω_0.

In Section 2.2, this formulation is used to study well-posedness and the blow-up of solutions in two and three dimensions.

In the following sections, we return to the velocity and vorticity formulations and discuss stationary solutions and review stability results. There our focus is on the two-dimensional case.

1.5. Shear flow solutions to the 2D Euler equations. In this section, we introduce shear flow solutions to the 2D Euler's equations and briefly discuss the structure of stationary solutions.

Lemma 1.4 ([**Swa00**, page 93]; [**Zil10**, section 4]). *Let (ω, ϕ) be a (regular, classical) solution of the vorticity-stream formulation of Euler's equations. Then ω is a stationary solution, if and only if $\nabla\phi$ and $\nabla\Delta\phi$ are collinear.*

Proof. By the Euler equations, ω is a stationary solution if and only if

$$0 = v \cdot \nabla\omega = \nabla^{\perp}\phi \cdot \nabla\Delta\phi,$$

where we used that $\omega = \Delta\phi$. As we are in $\mathbb{R}^2$

$$\nabla^{\perp}\phi = (-\partial_y\phi, \partial_x\phi)$$

is obtained from $\nabla\phi$ by a rotation by $\frac{\pi}{2}$ and is thus orthogonal to $\nabla\Delta\phi$ if and only if $\nabla\phi$ and $\nabla\Delta\phi$ are parallel. $\square$

As the gradient is the normal of the level set, the preceding result also has implications for the level sets of ϕ and $\omega = \Delta\phi$. We, however, remark that in the case of a vanishing gradient the identification of level sets degenerates in the sense that, e.g. for the pair (ϕ, ω) with

$$\phi = \frac{y^2}{2},$$
$$\omega = 1,$$

ω is constant on every level set of ϕ, but the converse does not hold. Assuming some additional non-degeneracy, i.e. that $\nabla\omega$ is non-trivial, locally the converse holds and thus there locally exists a function F such that

$$\phi = F(\Delta\phi).$$

At this point, we introduce some notation for the formulation of Arnold's stability theorem in Section 2.3.

Definition 1.3 ([**Zil10**], section 4). Let ϕ be a stationary solution and let thus $\nabla\phi$ and $\nabla\Delta\phi$ be collinear. Then on the set where $\nabla\Delta\phi \neq 0$, there exists λ such that

$$\nabla\phi = \lambda\nabla\Delta\phi.$$

We follow Arnold's notation and denote

$$\frac{\nabla\phi}{\nabla\Delta\phi} := \lambda.$$

COROLLARY 1.1 ([**Zil10**], section 4). *Let $\phi \in C^3$ be a stationary solution and let F be such that locally*

$$\phi = F(\Delta\phi),$$

and $\nabla\Delta\phi \neq 0$. Then,

$$F'(\Delta\phi) = \frac{\nabla\phi}{\nabla\Delta\phi}.$$

PROOF. By the chain rule

$$\nabla\phi = F'(\Delta\phi)\nabla\Delta\phi,$$

which agrees with our definition of

$$\frac{\nabla\phi}{\nabla\Delta\phi}.$$

□

A particular class of stationary solutions is given by those whose stream function, ϕ, depends only on a single variable, e.g. $\phi = \phi(y)$. In this case

$$\nabla\phi \parallel e_2 \parallel \nabla\Delta\phi.$$

Solutions of this form are called shear flows.

DEFINITION 1.4 (Shear flow). Let $U(y)$ be a smooth function. Then the stationary solution of Euler's equations given by

$$\begin{aligned} v &= (U(y), 0), \\ p &= 0, \end{aligned}$$

is called a *shear flow* and $U(y)$ is called its profile.

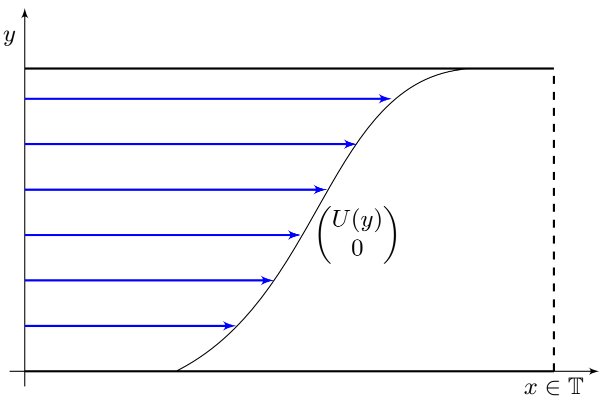


FIGURE 1. A shear flow $(U(y), 0)$ in a finite periodic channel

The behavior of solutions to Euler's equation close to shear flows is the main topic of this thesis.

1.6. Linearization around shear flows. Let $U(y) \in C^2$ be the profile of a shear flow and consider solutions of Euler's equation of the form $\omega = -U' + \epsilon\omega^*, v = (U(y), 0) + \epsilon v^*$.

Then they solve

$$\begin{aligned}\epsilon(\partial_t \omega^* + U(y)\partial_x \omega^* - v_2^* U'') &= \epsilon^2 v^* \cdot \nabla \omega^*, \\ v^* &= \nabla^\perp \Delta^{-1} \omega^*.\end{aligned}$$

Considering very small perturbations, i.e. ϵ very small, we linearize and hence omit the right-hand-side.

For convenience of notation, in the sequel we further omit the $*$ and denote the perturbation by ω, v, if there is no danger of confusion.

Definition 1.5 (Linearized Euler equations). The linearized Euler equations around a shear flow $(U(y), 0)$ are given by

$$\begin{aligned}\partial_t \omega + U(y)\partial_x \omega &= U'' v_2 = U'' \partial_x \phi, \\ \Delta \phi &= \omega,\end{aligned} \tag{E}$$

where the boundary conditions on ϕ are chosen depending on the domain considered.

As suggested by our notation, we consider the transport on the left-hand-side

$$\partial_t + U(y)\partial_x$$

and the resulting shearing as the main underlying dynamics and the right-hand-side as a perturbation. The heuristic implications of this perspective are discussed in the following Chapter 2.

Before that, we recall the main classical stability results for Euler's equations and in particular their implications for the stability and long-time asymptotic behavior of shear flows.

1.7. Couette flow. A particularly simple shear flow is given by $U(y) = y$. This flow is called Couette flow and is distinguished from other shear flows by also being a stationary solution of the Navier-Stokes equation. Furthermore the linearized equations around Couette flow are greatly simplified as U'' identically vanishes.

The linearized Euler equations are thus given by

$$\begin{aligned}\partial_t \omega + y\partial_x \omega &= 0, \\ v &= \nabla^\perp \Delta^{-1} \omega.\end{aligned} \tag{Couette}$$

We in particular note that the first equation does not contain v and that, in the case of an infinite periodic channel, $(x, y) \in \mathbb{T} \times \mathbb{R}$, the first equation is identical (up to notation) to the free transport equation.

As we discuss in Section 3, the study of this flow has a long history starting from the work of Kelvin, [**Kel87**], and Orr, [**Orr07**]. Many fascinating problems concerning its asymptotic behavior and stability are associated with it.

In particular, it has been observed that the velocity field of small perturbations asymptotically decays with algebraic rates (in L^2), a phenomenon which is called (linear) *inviscid damping*. While this linear result for Couette flow is classical and admits explicit solutions, as we discuss in Chapter 2, extending the result to other shear flows or the nonlinear equation has remained open until recently:

- The present work, for the first time, rigorously establishes linear inviscid damping with optimal algebraic rates and asymptotic stability in Sobolev spaces for a large class of monotone shear flows, of which Couette flow is one prototypical example.

- First results on nonlinear inviscid damping for Couette flow have recently been obtained by Masmoudi and Bedrossian [**BM13b**] and are discussed in Chapter 6.

2. Stability and well-posedness

2.1. Classical linear stability. As with any evolution equation a prominent question concerns the long-time dynamics and stability of solutions. Since the equations are nonlinear however, it is difficult to give meaningful general answers. Therefore much attention has been payed to special cases and linearizations around specific flows, in particular shear flows, and specific forms of perturbations.

Here the classical results are due to Rayleigh [**Ray79**] and were later extended by Fjørtoft (see [**DR04**], [**Dra02**]). Both consider so-called normal-mode perturbations to the linearized Euler equations around shear flows. In the sequel we briefly discuss these results.

Considering solutions close to a given shear flow $(U(y), 0)$, i.e. a vorticity of the form $\Omega = -U' + \omega$, the linearized Euler equations are given by

$$\begin{aligned} \partial_t \omega + U\partial_x \omega &= U''\partial_x \phi, \\ \Delta \phi &= \omega. \end{aligned}$$

As U, U'' do not depend on x, in the case of an infinitely long or periodic channel, i.e. in the cases of $x \in \mathbb{R}$ or $x \in \mathbb{T}$, we may take a Fourier transform in x and thus obtain the following decoupled system of equations for each frequency k

$$\begin{aligned} \partial_t \hat{\omega} + ikU\hat{\omega} &= U''ik\hat{\phi}, \\ (-k^2 + \partial_y^2)\hat{\phi} &= \hat{\omega}. \end{aligned} \tag{E}$$

For convenience, in following we drop the hats, $\hat{\cdot}$, from our notation, if there is no danger of confusion.

In this setting Rayleigh and Fjørtoft study the question of exponential instability under *normal-mode perturbations*. More precisely, they ask whether there exist non-trivial solutions with a stream function of the form

$$\phi(t, x, y) = f(y)e^{i\tau t + ikx},$$

where $k \in \mathbb{R} \setminus \{0\}, \tau \in \mathbb{C}, \Im(\tau) < 0$ and $f \in L^2(\mathbb{R})$ or $f \in C_0^2([0, 1])$ in the case of a finite channel (the boundary conditions correspond to impermeable walls). This type of instability is called *exponential* or *spectral instability*.

Using this ansatz, the ODE satisfied by f is then given by

$$\begin{aligned} i\tau(f'' - k^2 f) + ikU(f'' - k^2 f) &= U''ikf. \\ \Leftrightarrow (i\tau + ikU)f'' + (-ik^2\tau - ik^3U - ikU'')f &= 0. \end{aligned}$$

Dividing by ik and denoting the phase velocity by $c = -\frac{\tau}{k}$, we arrive at

$$\begin{aligned} (U - c)f'' + (k^2c - k^2U - U'')f &= 0, \\ \Leftrightarrow (U - c)(f'' - k^2 f) - U''f &= 0. \end{aligned} \tag{2}$$

The theorems of Rayleigh and Fjørtoft provide necessary conditions for the existence of non-trivial solution (f, c, k) with $\Im(\tau) = -k\Im(c) < 0$ to (2). In particular, if these conditions are not satisfied, no exponential instability in the above sense is possible, which can be interpreted as a stability result.

REMARK 1. *If there exists an exponentially decaying solution (f, c, k), then $(\overline{f}, \overline{c}, -k)$ is also a solution and is exponentially increasing. Here, $\bar{\cdot}$ denotes complex conjugation.*

This symmetry corresponds to the time-reversibility of the equation, i.e. an exponentially decreasing solutions blows up as $t \to -\infty$. In particular, if $c \notin \mathbb{R}$, then either c or $\overline{c}$ corresponds to a exponentially growing solution. Stability thus requires that there are no non-trivial solutions for $c \notin \mathbb{R}$.

THEOREM 1.1 (Rayleigh's theorem [**DR04**, page 131]). *A necessary condition for spectral instability of a shear flow $(U(y), 0)$ is that U'' changes sign or vanishes on a set of positive measure.*

PROOF. Suppose there exists $c \in \mathbb{C}$ with $\Im(c) < 0$ and a non-trivial solution f decaying at infinity. As $c \notin \mathbb{R}$, the quotient $\frac{U''}{U-c}$ exists and is bounded if U'' is bounded.

Testing

$$f'' - k^2 f + \frac{U''}{U-c} f = 0 \tag{3}$$

with $\overline{f}$ and considering only the imaginary part, we obtain

$$\int \Im\left(\frac{U''}{U-c}\right) |f|^2 = 0.$$

Computing

$$\Im\left(\frac{U''}{U-c}\right) = U'' \Im(c) \frac{1}{|U-c|^2},$$

and recalling that $|f|^2 \geq 0$ is assumed to be non-trivial, this leads to a contradiction unless U'' vanishes on the support of f or changes sign. □

Fjørtoft further improves this theorem by making use of the real part.

THEOREM 1.2 (Fjørtoft's theorem [**DR04**, page 132]). *Another necessary condition for spectral instability of the shear flow $(U(y), 0)$, is that for any point y_s with $U''(y_s) = 0$, the inequality*

$$U''(y)(U(y) - U(y_s)) < 0$$

holds for some y.

PROOF. Let $c \in \mathbb{C}$ with $\Im(c) < 0$ and suppose that there exists a non-trivial solution f decaying at infinity. Testing (3) with $\overline{f}$ again, but now considering the real part, we obtain

$$\int \frac{U''(U - \Re(c))}{|U-c|^2} |f|^2 = -\int |f'|^2 + k^2 |f|^2 < 0.$$

Using the proof of the previous theorem, we further know that

$$\int \frac{U''}{|U-c|^2} |f|^2 = 0.$$

Subtracting a suitable multiple of the second equation from the first, thus yields

$$\int U''(U-d) \frac{|f|^2}{|U-c|^2} < 0,$$

for any $d \in \mathbb{R}$. This, however, can only be the case if $U''(y)(U(y) - d) < 0$ for some y. Choosing d appropriately concludes the proof. □

We again stress that the previous theorems do not prove linear stability in the classical sense, but rather give a sufficient but not necessary condition to rule out a specific type of linear instability. In Section 1 of Chapter 6 we show that a large class of monotone shear flows is spectrally stable regardless of inflection points.

In the following sections, we consider well-posedness and nonlinear (Lyapunov) stability.

2.2. The Beale-Kato-Majda criterion. As discussed in Section 1.4, the Euler equations in two and three dimensions can, for sufficiently regular flows, equivalently be formulated as equations for the flow map

$$\frac{d}{dt}X(t,\alpha) = v(X(t,\alpha),t),$$
$$X(t,\alpha) = \alpha.$$

Heuristically, one would thus expect local existence if v is Lipschitz continuous.

Beale, Kato and Majda, [**BKM84**], prove an even better result and show that it suffices to control

- $\omega = \nabla \times v$ instead of the full gradient ∇v and that
- the L^∞ control need only hold in an integrated sense, i.e.

$$\int_0^T \|\omega(t)\|_{L_x^\infty} dt < \infty.$$

Under these conditions, they prove local existence and that, indeed, this is the only way local existence can fail. Thus, assuming a good control on

$$\int_0^T \|\omega(t)\|_{L_x^\infty} dt,$$

the local existence result can be iterated to yield global existence.

THEOREM 1.3 ([**MB01**, page 146], [**BKM84**]). *Let $\omega_0 = \nabla \times v_0$, with $\nabla \cdot v_0 = 0$, be compactly supported and suppose that $\omega_0 \in C^\gamma$ for some $\gamma > 0$. Suppose that for any time $T > 0$, there exists M_1 such that*

$$\int_0^T \|\omega(\cdot,s)\|_{L_x^\infty} ds \leq M_1,$$

then the solution exists globally in time.

Conversely suppose that there exists a maximal time of existence $T^\star < \infty$, then necessarily

$$\lim_{T\to T^\star} \int_0^T \|\omega(\cdot,s)\|_{L_x^\infty} ds = \infty.$$

In three dimensions, due to the vortex-stretching term in the evolution, obtaining a bound on $\|\omega\|_{L^1_{t,loc}L_x^\infty}$ is a challenging problem. In contrast, *in two dimensions*, the vorticity ω_0 is transported and thus the L^∞ norm at any given time $T > 0$ (formally) equals its initial value $\|\omega_0\|_{L^\infty}$. In order to make use of such conserved quantities, however, as shown by the results of Buckmaster, De Lellis, Székelyhidi [**BDLSJ14**], some regularity requirement is necessary. Under suitable assumptions, the previous theorem thus yields global existence for the 2D Euler equations.

COROLLARY 1.2 (2D global existence). *Let $\omega_0 = \nabla \times v_0, \nabla \cdot v_0 = 0$ be compactly supported and suppose that $\omega_0 \in C_b^\gamma$ for some $\gamma > 0$. Then there exists a global solution.*

PROOF OF COROLLARY 1.2. Suppose to the contrary, that for a given ω_0, there exists a maximal time of existence $0 < T^\star < \infty$.

Then, for any $t \in [0, T^\star)$, as shown in Section 1.4, ω satisfies

$$\omega(t, X(t,\alpha)) = \omega_0(\alpha).$$

Hence, in particular, for all such t

$$\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}.$$

Integrating this equality, we obtain

$$\lim_{T \to T^\star} \int_0^T \|\omega(t)\|_{L^\infty_{x,y}} dt = \|\omega_0\|_{L^\infty_{x,y}} T^\star < \infty.$$

Therefore, by the preceding theorem, $T^\star$ is not maximal. Contradiction. $\square$

We further remark, that *in two dimensions* there also exist results on uniqueness and continuity of the solution map. Therefore, this result implies global well-posedness for the 2D Euler equations in L^∞ with some additional regularity assumptions.

2.3. Nonlinear stability and Hamiltonian structure. In [**Arn66b**], Arnold gives a characterization of Euler's equations on a Riemannian manifold M as the geodesic equations on the (infinite-dimensional) manifold of smooth volume-preserving diffeomorphisms, $\mathrm{SDiff}(M)$.

For simplicity, we here only provide a brief sketch of these results for the case of $M = \mathbb{R}^2$ or $M = \mathbb{R}^3$. A more thorough discussion can be found in the books of Arnold, Khesin and Wendt [**AK98**], [**Arn89**], [**KW09**] as well as the author's Bachelor thesis [**Zil10**].

As we have seen in Section 1.4, under smoothness assumptions, Euler's equations can be alternatively understood as equations for the flow map $X(t, \alpha)$. By the incompressibility condition, these maps are volume-preserving and thus invertible. We will additionally assume that they are C^∞. The set of all such diffeomorphisms then has the structure of Fréchet manifold over $C^\infty(M; M)$. Furthermore, the diffeomorphisms have a group structure under composition and the group operations are smooth. We hence obtain a Fréchet Lie group.

An energy functional E can then be defined by the kinetic energy

$$E(v) = \frac{1}{2} \int_M v^2.$$

This functional is right-invariant, since composition by a volume-preserving map preserves the L^2 scalar product and yields a Riemannian structure on the Lie group and Lie algebra by identification.

Arnold explicitly computes the geodesic equations with respect to this Riemannian structure in terms of group operations and the Lie bracket, which in this particular case corresponds to a commutator of vector fields.

The geodesic equations are given by

$$\partial_t v = -B(v, v),$$

where B is defined via the commutator of vector fields, $[\cdot, \cdot]$, as

$$\langle B(c, a), b\rangle = \langle [a, b], c\rangle.$$

PROPOSITION 1.5 ([**AK98**, page 20]). *In the case of 3D hydrodynamics, i.e. $M = \mathbb{R}^3$, the geodesic equations are given by*

$$\partial_t v = v \times (\nabla \times v) - \nabla p,$$

which is equivalent to Euler's equations.

PROOF. Let a, b, c be divergence-free vector fields, then

$$\langle B(c, a), b\rangle = \langle \nabla \times (a \times b), c\rangle = -\langle a \times b, \nabla \times c\rangle = -\langle (\nabla \times c) \times a, b\rangle.$$

Thus, $B(v, v) = v \times (\nabla \times v) + \nabla p$ for some pressure p, as b is divergence-free.

In order to see that this equation is equivalent to the velocity formulation of Euler's equations, we compute

$$\begin{aligned}(v \times (\nabla \times v))_k &= \epsilon_{ijk} v_i (\nabla \times v)_j = \epsilon_{ijk} v_i \epsilon_{lmj} \partial_l v_m \\ &= -(\epsilon_{ikj}\epsilon_{lmj}) v_i \partial_l v_m = -(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) v_i \partial_l v_m \\ &= -(v \cdot \nabla) v_k + \partial_k |v|^2.\end{aligned}$$

Modifying $p \mapsto p + |v|^2$, hence shows the equivalence. □

Interpreting Euler's equations as geodesic equations, allows one to use methods of geometry and the calculus of variations to study stability. As we are only permitted to vary via group actions, it however has to be noted that the group action of volume-preserving diffeomorphisms is not surjective. For example, in the 2D case,

$$\omega \mapsto \omega \circ X$$

preserves all L^p norms, for any vorticity ω and any volume-preserving diffeomorphism X (not necessarily a solution). Hence, in this case, any two vorticities ω_1, ω_2 for which for some $p \in [1, \infty]$

$$\|\omega_1\|_{L^p} \neq \|\omega_2\|_{L^p},$$

can not share the same orbit, as $\omega_1 \circ X_1 = \omega_2 \circ X_2$ would imply

$$\|\omega_1\|_{L^p} = \|\omega_1 \circ X_1\|_{L^p} = \|\omega_2 \circ X_2\|_{L^p} = \|\omega_2\|_{L^p},$$

and thus yield a contradiction.

For the purpose of variational arguments, it is thus necessary to restrict to the orbit of a given vorticity under the group action. Vorticities with the same orbit are called *isovortical*. Using this as an equivalence relation, we obtain a foliation of L^2. It can then be shown that stationary solutions are distinguished points on each leaf.

Lemma 1.5 ([**AK98**]; see also [**Zil10**]). *Consider a leaf of the isovortical foliation, then a point is a stationary solution, if and only if it is a critical point of the energy functional restricted to this leaf.*

Sketch of proof. Let v be a given point and ξ a tangent vector at that point. It can be shown that the bilinear form $B(\cdot,\cdot)$ from above is non-degenerate. As a consequence, there exists f such that

$$\xi = B(v, f).$$

The variation of the energy at v in direction ξ is hence given by

$$\delta E = \langle v, \xi \rangle = \langle v, B(v, f) \rangle = -\langle B(v, v), f \rangle.$$

δE therefore vanishes for all f, if and only if $B(v, v) = 0$, which is equivalent to the solution being stationary. □

Considering the second variation in such critical points, one can obtain stability results, if the second variation is positive definite. This is in analogy to the finite-dimensional situation.

Theorem 1.4 ([**AK98**, page 90]; see also [**Zil10**]). *Let v be a stationary solution with stream function ψ. Then the second variation of the energy restricted to the corresponding leaf is given by*

$$\delta^2 E = \frac{1}{2} \int (\delta v)^2 + \frac{\nabla \psi}{\nabla \Delta \psi} (\delta \omega)^2,$$

where $\delta v, \delta \omega$ is the velocity and vorticity perturbation.

Recall that the notation

$$\frac{\nabla\psi}{\nabla\Delta\psi}$$

has been introduced in Section 1.5. Supposing a good control on this quotient, Arnold then establishes a stability theorem.

THEOREM 1.5 (Arnold's stability theorem [**Arn66b**], [**Arn89**], [**AK98**]; see also [**Zil10**], [**Zil12**]). *Let ψ be the stream function of a stationary solution. Suppose further that $\phi = F(\Delta\phi)$ globally and that*

$$0 < c < \frac{\nabla\psi}{\nabla\Delta\psi} < C < \infty.$$

Then any solution with stream function $\psi + \phi$ with the same circulation around the boundary, satisfies

$$\int |\nabla\phi|^2 + c|\Delta\phi|^2 \leq \int |\nabla\phi_0|^2 + C|\Delta\phi_0|^2.$$

Similarly, if

$$0 < c < -\frac{\nabla\psi}{\nabla\Delta\psi} < C < \infty,$$

then any solution with stream function $\psi + \phi$ with the same circulation around the boundary, satisfies

$$\int c|\Delta\phi|^2 - |\nabla\phi|^2 \leq \int C|\Delta\phi_0|^2 - |\nabla\phi_0|^2.$$

In the first case, Arnold's theorem provides a control of the perturbation's (change to the) kinetic energy and enstrophy

$$\|\nabla^\perp\phi\|_{L^2}^2 + \|\Delta\phi\|_{L^2}^2.$$

It thus provides a *nonlinear stability result.* However, as we discuss in the sketch of the proof, it crucially relies on a convexity mechanism via the control of the quotient $\frac{\nabla\phi}{\nabla\Delta\phi}$ as well as a conserved quantity. In particular, for most monotone shear flows $(U(y), 0)$, including Couette flow,

$$\frac{\nabla\phi}{\nabla\Delta\phi} = \frac{U}{U''}$$

does not satisfy the assumptions of the theorem. For such flows, it is hence necessary to make use of a fundamentally different damping mechanism, called (linear) *invisid damping.* The study of this mechanism is the main topic of the present thesis and is introduced on a heuristic level in Chapter 2.

SKETCH OF PROOF OF THEOREM 1.5. As remarked in the definition of $\frac{\nabla\psi}{\nabla\Delta\psi}$, by the chain rule

$$F'(\Delta\psi) = \frac{\nabla\psi}{\nabla\Delta\psi}.$$

Defining a primitive function G of (an extension of) F, by our assumptions G satisfies

$$c < G''(\Delta\phi) < C.$$

G is thus a convex function.

It can then be shown that

$$H_2(\phi) = \iint \frac{|\nabla\phi|^2}{2} + (G(\Delta\psi + \Delta\phi) - G(\Delta\psi) - G'(\Delta\psi)\Delta\phi)$$

is a conserved quantity.

By convexity and the mean value theorem, for any x and y

$$\frac{c}{2}y^2 \leq G(x+y) - G(x) - G'(x)y \leq \frac{C}{2}y^2.$$

Therefore,

$$\int |\nabla\phi|^2 + c|\Delta\phi|^2 \leq H_2(\phi) = H_2(\phi_0) \leq \int |\nabla\phi_0|^2 + C|\Delta\phi_0|^2.$$

The concave case, where

$$0 < c < -\frac{\nabla\psi}{\nabla\Delta\psi} < C < \infty,$$

is proven analogously. □

We remark that Arnold's theorem does not necessarily require that U is convex or concave but allows for inflection points. For example, considering the Kolmogorov flow $U(y) = \sin(y)$, we compute that

$$\frac{\sin(y)}{(\sin(y))''} \equiv -1,$$

where the singularity at the inflection points $y \in \pi\mathbb{Z}$ have been removed.

In the following section, we discuss conserved quantities of Euler's equations and their use in the study of stability.

2.4. Conserved quantities. As discussed in the previous section, Euler's equations can be interpreted as geodesic equations and thus in particular have the structure of a Hamiltonian system. Therefore, there exist many conserved quantities, of which we list several for the *two-dimensional* case. A more extensive list of conserved quantities, including the three-dimensional case, can be found in [**MB01**, section 1.7].

LEMMA 1.6. *Let ω be smooth solution of the 2D Euler's equations, then:*

- *The* kinetic energy
$$\|v(t)\|_{L^2}$$
is conserved.
- *The* enstrophy
$$\frac{1}{2}\|\omega(t)\|_{L^2}$$
is conserved.
- *Let $f \in C^0$, then*
$$\int f(\omega)$$
is conserved. In particular, for any non-negative vorticity the entropy
$$\int \omega \log(\omega)$$
is conserved.

The last family of conserved quantities is very general and includes not only all $L^p, p \in [1,\infty]$ norms by choosing

$$f(x) = |x|^p,$$

but also the *entropy*

$$f(x) = x\log(x)$$

for non-negative vorticities.

The Euler equations thus exhibit neither dissipation nor entropy increase or other usual damping mechanisms and are even time-reversible.

Furthermore the conserved quantities imply nonlinear L^p stability of Couette flow, i.e. $v(t,x,y) = (y,0)$, for any $p \in [1,\infty]$.

Lemma 1.7. *Let $v = (y,0,) + v'$, $\omega = -1 + \omega'$ be a solution of the 2D Euler equations. Then for any $p \in [1,\infty]$ and any $t \in \mathbb{R}$*

$$\|\omega'(t)\|_{L^p} \equiv \|\omega'|_{t=0}\|_{L^p}.$$

Proof. By the particle-trajectory formulation, at any time t

$$\omega = \omega|_{t=0} \circ X = 1 + \omega'|_{t=0} \circ X$$

for some volume preserving diffeomorphism X. Thus, for any $p < \infty$, choosing

$$f(x) = |x-1|^p$$

in the previous lemma, the L^p norm is conserved. Similarly the L^∞ norm is conserved. $\square$

We remark that this result crucially uses that the vorticity corresponding to Couette flow is constant. Conversely any shear flow with constant vorticity is affine and thus equal to Couette flow up to symmetries. While very powerful, this result is hence not useful for considering perturbations around any more general shear flows.

When considering the linearized problem instead, using a condition similar to Arnold's, we can however still make use of conserved quantities to derive a stability result.

Lemma 1.8 ([**Zil12**, page 30]). *Let $U(y)$ be a shear profile such that*

$$\frac{U}{U''}$$

is well-defined. Then for any solution (ω, v) of the linearized equations

$$\int v^2 + \frac{U}{U''}\omega^2$$

is conserved.

We stress that, unlike Arnold's stability theorem, this only provides *linear stability*. Furthermore, for most monotone shear flows, including Couette flow, the quotient $\frac{U}{U''}$ does not satisfy the assumptions of the previous lemma and thus a different stability mechanism has to be used.

In the following sections we introduce the Vlasov-Poisson equations of plasma physics and discuss their structural similarities with Euler's equations. As we discuss in Chapters 2 and 3, for these equations one encounters the phenomenon of *Landau damping*, which motivates our study of linear inviscid damping for general monotone shear flows.

3. The Vlasov-Poisson equations

In this section, we briefly introduce the Vlasov-Poisson equations of plasma physics and their homogeneous solutions. Subsequently, we discuss the structural similarities with the Euler equations in Section 4.

The Vlasov-Poisson equations model the evolution of the phase-space density $f(x,v,t)$,

$$f : \mathbb{T}^n \times \mathbb{R}^n \times (0,T) \to [0,\infty)$$

of a plasma and are given by

$$\partial_t f + v \cdot \nabla_x f + F[f](t,x) \cdot \nabla_v f = 0,$$
$$F[f] = \nabla W *_x \left(\int f dv - \iint f dv dx \right) := \nabla W *_x (\rho - \langle \rho \rangle_x).$$

We stress that, unlike in the previous setting of Euler's equations, in the context of the Vlasov-Poisson equations and other transport equations $v \in \mathbb{R}^n$ is used to denote a variable instead of a vector field.

$F[f]$ is a force field and depends on f only via the density

$$\rho(x,t) = \int_{\mathbb{R}^n} f(x,v,t) dv.$$

Various choices for W are possible, but in the following we focus on the cases of a Coulomb or Newton potential, i.e.

$$F[f](x,t) = \pm \nabla_x \Delta_x^{-1} (\rho - \langle \rho \rangle_x).$$

Formally, one can see that

$$\nabla W *_x c = W *_x \nabla c = W *_x 0 = 0$$

for any constant c and therefore one might be tempted to remove $\langle \rho \rangle_x$ in the above formulae. However, as ∇W is not in L^1, this heuristic is not quite rigorous and is known as the so-called *Jeans swindle*. A rigorous justification for this “swindle” can be obtained by defining F as the limiting case of some cut-off potentials W_ϵ for which $\nabla W_\epsilon \in L^1$ (confer [**MV10a**] and the references therein).

By the above reasoning, we, however, see that densities f with

$$\rho - \langle \rho \rangle_x \equiv 0,$$

play a distinct role. A particular class of such solutions is given by *homogeneous solutions*, i.e.

$$f = f(v)$$

being independent of x.

As the force field vanishes for such solutions, it seems reasonable to assume that solutions close to homogeneous solutions behave similar to solutions of free transport. The implications of these heuristics are discussed in more detail in Section 4 of Chapter 2.

3.1. Linearization around homogeneous solutions. In this section, we introduce the linearization of the Vlasov-Poisson equations around homogeneous solutions.

Let $f(t,x,v) = f_{in}(v) + \epsilon h(t,x,v)$ be a solution of the Vlasov-Poisson equation. Then h satisfies

$$\epsilon \partial_t h + \epsilon v \cdot \nabla_x h + \epsilon F[h] \cdot \nabla_v f_{in} = -\epsilon^2 F[h] \nabla_v h.$$

Here we used that

$$F[f_{in} + \epsilon h] = F[f_{in}] + \epsilon F[h] = \epsilon F[h].$$

Neglecting the quadratic nonlinearity, we arrive at the linearized Vlasov-Poisson equations.

DEFINITION 1.6 (Linearized Vlasov-Poisson equations). Let $f_{in}(v)$ be a homogeneous density, then the *linearized Vlasov-Poisson equations* around f_{in} are given by

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h &= -F[h] \cdot \nabla_v f_{in}, \\ F[h] &= \nabla W *_x (\rho - \langle \rho \rangle_x), \\ \rho(t,x) &= \int h(t,x,v) dv. \end{aligned} \tag{LVP}$$

It has been shown by Landau, [**Lan46**], that, under certain condition on f_{in}, W and $h|_{t=0}$, one observes *Landau damping*, i.e. the force field decays in time. A heuristic for the underlying mechanism of Landau damping is given in Chapter 2. Subsequently, in Chapter 3, we briefly sketch a proof of linear Landau damping and discuss Villani and Mouhot's seminal results, [**MV11**], on nonlinear Landau damping.

In the following section, we discuss the structural similarities of the Vlasov-Poisson equations and the 2D Euler equations.

4. The connections between Euler and Vlasov-Poisson

In this section, we discuss the structural similarities and differences of the Vlasov-Poisson and Euler's equations, following the book of Majda and Bertozzi [**MB01**, chapter 13] as a reference.

As discussed in the previous section, the Vlasov-Poisson equation are given by

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + F[f](t,x) \cdot \nabla_v f = 0, \\ F[f] = \nabla W *_x (\rho - \langle \rho \rangle_x), \\ \rho = \int f dv, \end{aligned}$$

where, for this section, we restrict to the one-dimensional case, i.e.

$$f : \mathbb{T} \times \mathbb{R} \times (0,T) \to [0,\infty).$$

Furthermore, we restrict F to the case of Coulomb or Newton interaction:

$$F[f] = \pm \partial_x (\partial_x^2)^{-1} (\rho - \langle \rho \rangle_x).$$

We remark that for Vlasov-Poisson equations the underlying domain has the structure of a tangent bundle, e.g. $\mathbb{T} \times \mathbb{R} = T\mathbb{T}$. Hence, in contrast to the Euler setting, geometric settings with boundary effects such as a finite periodic channel, $\mathbb{T} \times [0,1]$, are not physically relevant.

Denoting

$$\psi = \Delta^{-1}(\rho - \langle \rho \rangle_x) = \partial_{xx}^{-1}(\rho - \langle \rho \rangle),$$

and further introducing

$$\phi = \frac{v^2}{2} \pm \psi,$$

we may express our equation as

$$\begin{aligned} \partial_t f + u \cdot \nabla_{x,v} f &= 0, \\ \nabla \cdot u &= 0, \\ u = (v, -\partial_x \psi) &= \nabla^\perp \phi, \\ \nabla \times u = \partial_{xx} \psi &= \rho - \langle \rho \rangle_x. \end{aligned}$$

As u is divergence-free, the associated flow is volume-preserving and

$$\langle \rho \rangle_x = \iint f$$

is a constant, which is preserved in time.

The Vlasov-Poisson equations are thus structurally very similar to the Euler equations in vorticity-stream formulation

$$\begin{aligned}
\partial_t \omega + v \cdot \nabla_{x,y} \omega &= 0, \\
\nabla \cdot v &= 0, \\
v &= \nabla^\perp \phi, \\
\nabla \times v = \Delta \phi &= \omega.
\end{aligned}$$

We however remark that there is a significant difference in the dependence of the potential ϕ on f or ω respectively: In the case of Vlasov-Poisson, ϕ depends on f via ρ only, the problem is thus effectively lower-dimensional (1D in this case).

Property	V-P	Euler
Nonlinear transport	$\partial_t f + u \cdot \nabla f = 0$	$\partial_t \omega + v \cdot \nabla \omega = 0$
	$\nabla \cdot u = 0$	$\nabla \cdot v = 0$
	$\nabla \times u = \int f dv - \langle \rho \rangle_x$	$\nabla \times v = \omega$
Stream function	$\phi = \frac{v^2}{2} + \Delta_x^{-1}(\int f dv - \langle \rho \rangle_x)$	$\phi = \Delta^{-1}\omega$
	$u = \nabla^\perp \phi$	$v = \nabla^\perp \phi$

FIGURE 2. Summary of the structural similarities, adapted from [**MB01**, page 512].

In addition to the above structure, both equations share further similarities such as in the study of electron sheets and vortex sheets. A more thorough discussion may be found in [**MB01**, chapter 13]. For our purposes, however, we are most interested in the similarities of the transport structure. In particular, considering homogeneous solutions of the Vlasov-Poisson equations, $f = f(v)$, or the Couette flow solution of Euler's equations in an infinite channel $\mathbb{T} \times \mathbb{R}$, the equations read:

$$\partial_t f + v \partial_x f = 0,$$

in the case of Vlasov-Poisson and

$$\partial_t \omega + y \partial_x \omega = 0,$$

in the case of Euler's equations. Recalling the difference in notation, i.e. $(x, v) \in \mathbb{T} \times \mathbb{R}$ instead of $(x, y) \in \mathbb{T} \times \mathbb{R}$, both equations are actually identical and are given by the free transport equation.

As we discuss in the following Chapter 3, this underlying transport structure plays an integral role in both Landau damping and inviscid damping. However, we again highlight the difference of the dependence of F on f and of the velocity field v on ω. As a consequence, the actual damping mechanism, the damping rates and the role of regularity differ.

CHAPTER 2

Free transport and phase-mixing

In this chapter we introduce the free transport equation and use its explicit solution to study its asymptotic behavior. While the free transport equation is time-reversible and possesses many conserved quantities and, in particular, exhibits neither dissipation nor entropy increase, one observes a very strong mixing behavior and resulting damping phenomena.

We discuss the *phase-mixing* mechanism underlying this damping and subsequently study the heuristic implications for the behavior of solutions of the Vlasov-Poisson equations

$$\partial_t f + v \cdot \nabla_x f = -F[f](t,x) \cdot \nabla_x f$$

close to homogeneous densities $f(v)$ and for the 2D Euler equations

$$\partial_t \omega + v \cdot \nabla \omega = 0$$

close to Couette flow, i.e. $v = (y, 0)$.

1. The free transport equation

The free transport equation models the evolution of a phase-space density transporting itself and is given by

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= 0, \\ f &= f(t,x,v), \\ (t,x,v) &\in \mathbb{R} \times \mathbb{T} \times \mathbb{R}. \end{aligned} \tag{4}$$

Using the method of characteristics, the solution of the free transport equation can be computed explicitly from its initial data

$$f(t,x,v) = f_0(x - tv, v).$$

We note that the equation is translation invariant with respect to x. Hence, even without a periodicity assumption, $x \in \mathbb{T}$, we could restrict to studying frequency-localized initial data. The periodicity assumption, however, provides a more natural frequency localization, a low-frequency cut-off and a better physical intuition.

In order to obtain first insights into the dynamics of general solutions to free transport on $\mathbb{T} \times \mathbb{R}$, it is instructive to consider the common example of f_0 being the characteristic function of a square, as depicted in Figure 1.

We observe that the solution is strongly sheared and mixed, which has two contrasting implications:

- The solution loses regularity in time and, unlike usual damping mechanisms such as dissipation, the behavior is very rough and "violent".
- Integral operators, averages or anti-derivatives benefit from the mixing behavior due to cancellations.

Reconciling these opposing behaviors and their effects on the regularity of solutions and potentials such as the velocity field $\nabla^{\perp}\Delta^{-1} f$, is at the core of both inviscid and

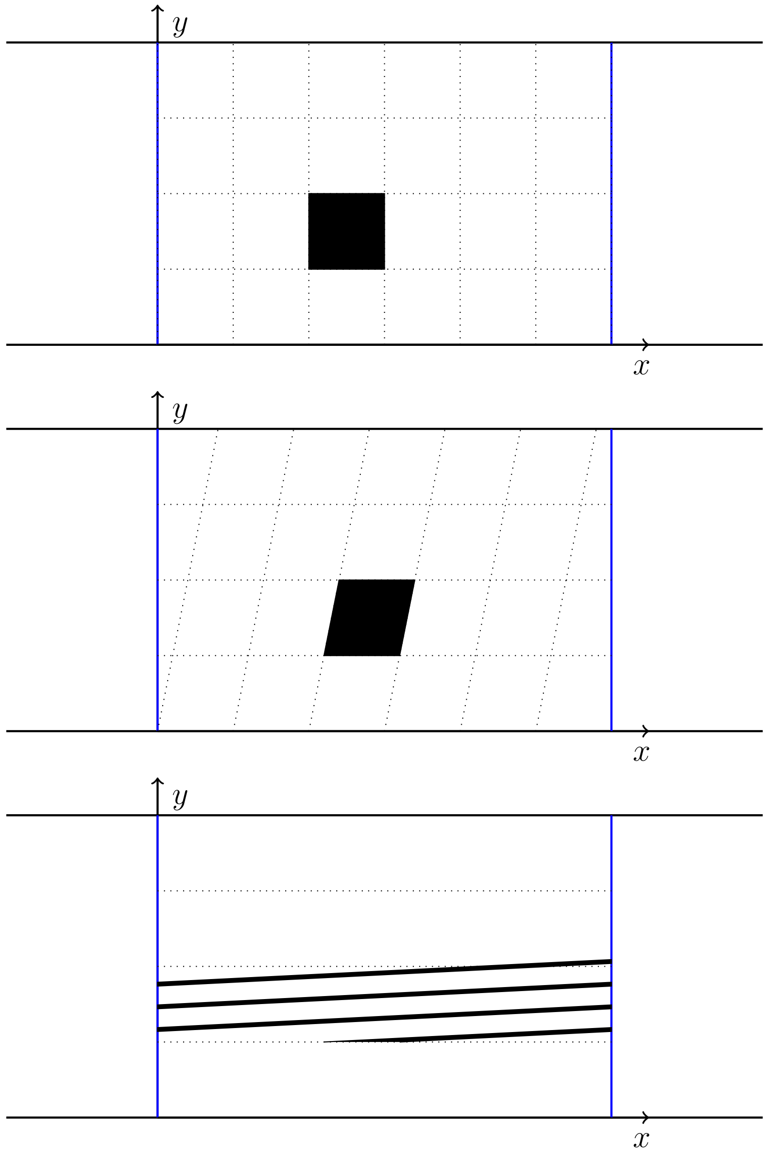


Figure 1. A characteristic function sheared by free transport at the initial time, after a small and a long time. See also [**Zil12**], [**MV10a**].

Landau damping. In the latter case, the mechanism is also referred to as *violent relaxation* (c.f. [**MV10a**]).

In the following sections, we study the explicit solution of free transport and a characterization in Fourier space in order to obtain a fine analysis of the damping mechanism and to deduce necessary requirements, which have to be imposed on the initial perturbation.

2. Transport, shearing and regularity

In this section, we discuss the effects of shearing on the regularity of solutions and, in particular, on potentials and negative Sobolev norms.

As shown in the following lemma, unlike dispersion, free transport preserves all L^p norms.

Lemma 2.1. *Let $p \in [1, \infty]$ and let f be a solution of the free transport equation with initial datum $f_0 \in L^p(\mathbb{T} \times \mathbb{R})$, then for any $t \in \mathbb{R}$*

$$\|f(t)\|_{L^p} = \|f_0\|_{L^p}.$$

Proof. As remarked previously, $f(t)$ can be computed explicitly in terms of f_0, using the method of characteristics:

$$f(t, x, v) = f_0(x - tv, v).$$

Hence, the L^p norm of f at a given time t satisfies

$$\|f(t)\|_{L^p}^p = \iint |f_0(x - tv, v)|^p dxdv = \iint |f_0(x', v)|^p dx'dv = \|f_0\|_{L^p}^p,$$

where we used that the change of variables

$$(x, v) \mapsto (x + tv, v)$$

is volume-preserving. □

We further note that, while different frequencies in x move with different speeds, e.g.

$$f_0(x) = \sin(kx) \mapsto f(t, x, v) = \sin(kx - ktv),$$

they do not asymptotically separate in space due to periodicity. They, however, separate in Fourier space, as we discuss in the following.

Let f be a solution of the free transport equation and denote its Fourier transform in both x and v by $\tilde{f}$. Then $\tilde{f}$ satisfies

$$\partial_t \tilde{f} - k\nabla_\eta \tilde{f} = 0, \tag{5}$$

where $k \in \mathbb{Z}$ and $\eta \in \mathbb{R}$ correspond to x and v respectively.

We observe that (5) is again of the form of a transport equation on $\mathbb{Z} \times \mathbb{R}$ and can be explicitly solved:

$$\tilde{f}(t, k, \eta) = \tilde{f}_0(k, \eta + kt).$$

Furthermore, the equation decouples with respect to k and we may hence treat $k \in \mathbb{Z}$ as a given parameter and separately consider the evolution of $f_k(t, \eta) := f(t, k, \eta)$.

Using the Fourier characterization, we obtain an explicit description of the evolution of arbitrary Sobolev norms.

Lemma 2.2. *Let f be a solution of free transport,* (5)*, with initial datum f_0. Then for any $s_1, s_2 \in \mathbb{R}$*

$$\begin{aligned}\|f(t)\|_{H^{s_1,s_2}}^2 :=& \sum_k \int (< k >^{2s_1} + < \eta >^{2s_2}) |\tilde{f}(t, k, \eta)|^2 d\eta \\ =& \sum_k \int (< k >^{2s_1} + < \eta - kt >^{2s_2}) |\tilde{f}_0(k, \eta)|^2 d\eta,\end{aligned}$$

where $< \cdot >$ is defined as

$$< x > := \sqrt{1 + |x|^2}.$$

Proof. We recall that $f(t)$ satisfies

$$\tilde{f}(t, k, \eta) = \tilde{f}_0(k, \eta + kt).$$

The result hence follows by a volume-preserving change of variables $\eta \mapsto \eta - kt$. □

As the equation decouples with respect to k, a finer description of the behavior can be obtained by considering single modes. More precisely, let $h(y)$ be a given (smooth, integrable) function and consider

$$f_0(x,y) = e^{ikx} h(y).$$

Then the solution f of the free transport equation with initial data f_0 satisfies

$$\|f(t)\|_{H^{0,s}} = \int (1+ <\eta - kt>^{2s}) |(\mathcal{F}_y h)(\eta)|^2 d\eta. \tag{6}$$

Denoting $\mathcal{F}_y h =: g$, in the following we thus study

$$I(t; g, k, s) := \int <\eta - kt>^{2s} |g(\eta)|^2 d\eta,$$

for given $k, s, t \in \mathbb{R}$. For simplicity, in the following we additionally assume that $g \in C_c^\infty(\mathbb{R})$ and $k > 0$.

The multiplier $<\eta - kt>^{2s}$ is depicted in Figure 2 for the case $k = 1$ and $s = -1$.

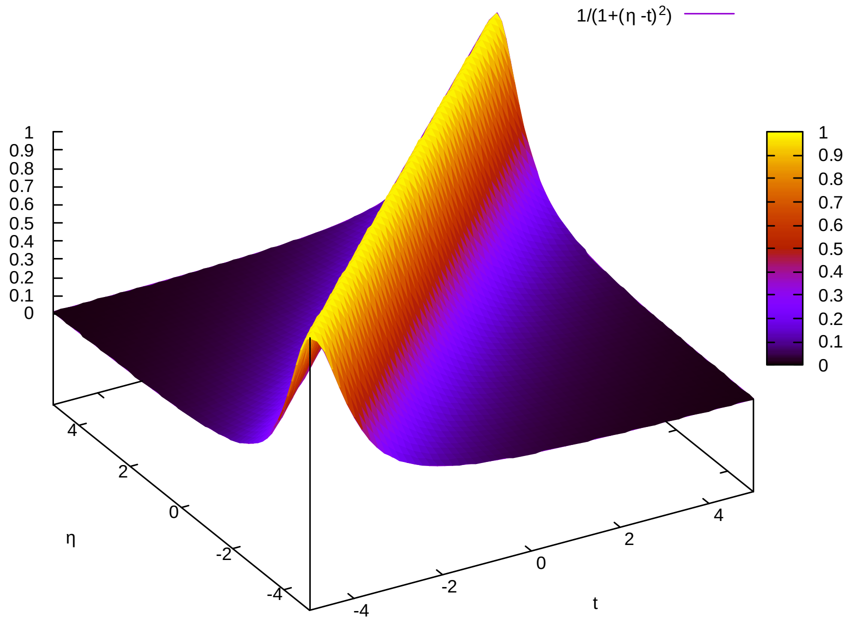


FIGURE 2. The multiplier, $<\eta - t>^{-2}$, decays as either t or η tend to infinity. This decay does, however, not hold on the diagonal $t = \eta$, where the multiplier equals 1.

Considering a fixed frequency (k, η), $k \neq 0$, we observe that

$$<\eta - kt>^{-2}$$

decays in t only after the *critical time* $t_c := \frac{\eta}{k}$ and, in fact, *increases* before the critical time. The implications of this change in behavior are discussed in the following lemmata and again in Section 3 in the context of the linearized 2D Euler equations around Couette flow.

Lemma 2.3. *Let $k < 0$ and suppose that $g \in C_c^\infty(\mathbb{R})$ is supported on the positive half-line, $\eta \geq 0$.*

Then, for any $s \geq 0$ and $t \in (0, \infty)$

$$I(0; g, k, s) + |kt|^{2s} \|g\|_{L^2}^2 \lesssim I(t; g, k, s) \lesssim I(0; g, k, s) + |kt|^{2s} \|g\|_{L^2}^2,$$

and $I(t; g, k, s)$ is monotonically increasing in t.

In the case $s \leq 0$ and $t \in (0, \infty)$

$$I(t; g, k, s) \lesssim \min(I(0; g, k, s), |kt|^s \|g\|_{L^2}^2)$$

and $I(t; g, k, s)$ is monotonically decreasing in t.

Proof. Let g, k and $t \geq 0$ be given. Then on the support of g all critical times $\frac{\eta}{k}$ are negative and hence

$$< \eta - kt > = < |\eta| + |k| t >$$

is monotonically increasing for $t \geq 0$. Furthermore

$$< \eta > + |kt| \lesssim < |\eta| + |k| t > \lesssim < \eta > + |kt|,$$

which yields the desired estimate.

Concerning the case $s \leq 0$, we note that negative powers of an increasing positive function are monotonically decreasing. □

In view of equation (6), the lemma hence agrees with the heuristics obtained from Figure 1, that negative Sobolev norms of f asymptotically decrease and positive Sobolev norms asymptotically increase. However, we stress that this is only the case asymptotically.

Lemma 2.4. *Let $k < 0$ and let $\eta < 0$, then for $t \in [0, \frac{\eta}{k}]$*

$$|\eta - kt|$$

is monotonically decreasing *in t.*

Hence, for any $s < 0$, $C > 1$ and any $T > 0$, there exists k and $g \in C_c^\infty(\mathbb{R})$ such that

$$\begin{aligned} I(0; g, k, s) &= 1, \\ I(t; g, k, s) \text{ is monotonically increasing in } t \text{ on } &[0, T], \\ I(T; g, k, s) &> C. \end{aligned}$$

Proof. The first statement follows by noting that for k, η, t as above,

$$|\eta - kt| = |\eta| - |k| t$$

is monotonically decreasing in $t \in [0, \frac{\eta}{k}]$.

It remains to prove the second statement. Let thus $s < 0$, $C > 1$ and $T > 0$ be given. Then there exist $k \in \mathbb{Z}$ and $\eta_0 \in \mathbb{R}$ such that

$$\begin{aligned} k &< 0, \\ \eta_0 &< 0, \\ \eta_0 &= kT, \\ < \eta_0 >^{-2s} &> C. \end{aligned}$$

Consider a function g, which is supported in a small neighborhood of η_0 and normalized such that

$$I(0; g, k, s) = 1.$$

Then, by the first result and our choice of k, η_0,

$$I(t; g, k, s) = \int < \eta - kt >^{2s} |g(\eta)|^2 d\eta$$

is monotonically increasing for $t \in (0, T)$. Furthermore,

$$
\begin{aligned}
I(T; g, k, s) &= \int < \eta - kT >^{2s} |g(\eta)|^2 d\eta \approx \int < \eta_0 - kT >^{2s} |g(\eta)|^2 d\eta \\
&= < \eta_0 >^{-2s} \int < \eta_0 >^{2s} |g(\eta)|^2 d\eta \\
&> C I(0; g, k, s) = C.
\end{aligned}
$$

□

Again recalling (6), we thus note that, while negative Sobolev norms of solutions, f, to free transport tend to zero asymptotically, they might grow by an arbitrarily large factor in an arbitrarily small time. However, for any $s < 0$ and any t

$$I(t; g, k, s) \leq I(t; g, k, 0) = \|g\|_{L^2}^2.$$

Hence, there can be no blow-up of $I(t; g, k, s)$ if $\|g\|_{L^2}^2$ is controlled. Recalling the definition of I via f_0, this can equivalently be expressed as

$$\|f(t)\|_{H^{0,s}} \leq \|f_0\|_{L^2_{x,y}},$$

for $s < 0$. Negative Sobolev norms of the solution to the free transport equation, (5), can thus be uniformly controlled by the *more regular* L^2 norm of the initial data.

In the context of the linearized Euler equations around Couette flow, the previously discussed growth of negative Sobolev norms, more precisely of the kinetic energy

$$\|v\|_{L^2} \approx \|\omega\|_{H^{-1}},$$

and the controlling influence of higher regularity is well-studied and known as the *Orr mechanism*. We discuss this in the following section.

3. Linear inviscid damping for Couette flow

As we have noted in Section 1.7, the *linearized* 2D Euler equations around Couette flow, i.e. around the shear flow $(y, 0)$, on an infinite channel $\mathbb{T} \times \mathbb{R}$

$$\partial_t \omega + y \partial_x \omega = 0,$$

are (up to a change of notation) identical to the free transport equation. Thus, by the discussion from the previous section, we expect perturbations to the velocity field, i.e.

$$v = \nabla^\perp \Delta^{-1} \omega,$$

to asymptotically decay (towards another shear flow).

Such dynamics for small perturbations have been experimentally observed and studied on a linearized level by Kelvin and Orr [**Kel87**], [**Orr07**] (see also [**LZ11**]). More precisely, as we discuss in the following for the setting of an infinite channel, it is shown that for solutions of the linearized equations with $\omega_0 \in H^2$:

- There exists a shear flow $(U(y), 0)$ such that

(7)
$$v - (U(y), 0) \xrightarrow{L^2} 0,$$

  as $t \to \infty$. That is,

$$\lim_{t \to \infty} \|v(t, \cdot) - (U, 0)\|_{L^2} = 0.$$

- The rate of convergence is algebraic:

$$
\begin{aligned}
\|v - (U(y), 0)\|_{L^2} &= \mathcal{O}(t^{-1}), \\
\|v_2\|_{L^2} &= \mathcal{O}(t^{-2}).
\end{aligned}
$$

This asymptotic convergence to another shear flow with algebraic rates is known as *(linear) inviscid damping.*

As the linearized Euler equations around Couette flow are identical to free transport, they, in particular, admit explicit solutions. Hence, in this case linear inviscid damping can be easily shown to hold via a direct computation. Going beyond the (in this sense trivial) setting of linear inviscid damping for Couette flow, however has remained mostly open until recently.

- In [**BM10**], Bouchet and Morita give heuristic results suggesting that linear damping and stability results should also hold for general monotone shear flows. However, their methods are highly non-rigorous and lack necessary regularity, stability and error estimates, as discussed in [**Zil12**]. In particular, even supposing their asymptotic computations were valid, they do not yield the above decay rates.
- Lin and Zeng [**LZ11**] use the explicit solution of linearized Couette flow to establish damping also in a finite periodic channel. Furthermore, they show the existence of non-trivial stationary solutions to the 2D Euler equations in arbitrarily small H^s neighborhoods of Couette flow for any $s < \frac{3}{2}$. As a consequence, nonlinear inviscid damping can not hold in such low regularity.
- Recently, following the work of Villani and Mouhot, [**MV11**], on nonlinear Landau damping, Masmoudi and Bedrossian, [**BM13b**], have proven nonlinear inviscid damping for small Gevrey (see Definition 3.1) perturbations to Couette flow in an infinite periodic channel. We briefly discuss their results and the additional challenges in the nonlinear setting in Chapter 6.

As the main result of this thesis, in Chapters 4 and 5, we, for the first time, rigorously establish linear inviscid damping for a general class of monotone shear flows. Here we treat both the setting of an infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, and a finite periodic channel $\mathbb{T} \times [0, 1]$ with impermeable walls. In the latter setting, we show that boundary effects play a non-negligible role and derive (almost) sharp results on the stability in fractional Sobolev spaces.

For simplicity, in this section, we discuss the linearized 2D Euler equations around Couette flow in an infinite periodic channel. This allows us to study the effects of the shearing and mixing by free transport on the evolution of the velocity field

$$\nabla^{\perp}\Delta^{-1}\omega,$$

via explicit solutions in both spatial and Fourier space.

More precisely, recall that the Fourier transform of the vorticity $\tilde{\omega}$, as a solution of free transport, satisfies

$$\tilde{\omega}(t, k, \eta) = \tilde{\omega}_0(k, \eta + kt).$$

Furthermore, $v = \nabla^{\perp}\Delta^{-1}\omega$ can be expressed in terms of ω using a Fourier multiplier:

$$\omega \mapsto v = \mathcal{F}^{-1}\begin{pmatrix} -\frac{i\eta}{k^2+\eta^2} \\ \frac{ik}{k^2+\eta^2} \end{pmatrix}\mathcal{F}\omega.$$

Combining both equations, we hence obtain an explicit expression for $\tilde{v}$:

$$\begin{aligned} \tilde{v}_1(t, k, \eta) &= \frac{i\eta}{k^2+\eta^2}\tilde{\omega}_0(k, \eta + kt), \\ \tilde{v}_2(t, k, \eta) &= \frac{ik}{k^2+\eta^2}\tilde{\omega}_0(k, \eta + kt). \end{aligned} \tag{8}$$

Using this explicit characterization, one can prove damping with algebraic rates and further show that the rates are optimal and that the regularity requirements on ω_0 are necessary.

LEMMA 2.5 (Compare [**LZ11**, Chapter 4]). *Let $\omega_0 \in L^2(\mathbb{T}\times\mathbb{R})$ be given and let v_1, v_2 satisfy* (8). *Then, as $t \to \infty$,*

(I) $v_2 \xrightarrow{L^2} 0$.

(II) $\tilde{v}_1(t,k,\eta) \to 0$ *for any $k \neq 0$ and almost every $\eta \in \mathbb{R}$. The average in x, $\tilde{v}_1(t,0,\eta)$, is time-independent.*

(III) *v converges to the shear flow $U(y) := \partial_y^{-1}\langle \omega_0\rangle_x$ in the sense that*

$$v - (U(y),0) \xrightarrow{L^2} 0.$$

(IV) *Suppose additionally that $\omega_0 \in H_x^{-1}H_y^1$, then*

$$\|v-(U(y),0)\|_{L^2} = \mathcal{O}(t^{-1})\|\omega_0\|_{H_x^{-1}H_y^1}.$$

(V) *Suppose additionally that $\omega_0 \in H_x^{-1}H_y^2$, then*

$$\|v_2\|_{L^2} = \mathcal{O}(t^{-2})\|\omega_0\|_{H_x^{-1}H_y^2}.$$

(VI) *These regularity requirements are necessary for uniform estimates and the decay rates are optimal, even under higher regularity assumptions.*

(VII) *Considering the velocity field moving with the flow $V(t,x,y) := v(t,x-ty,y)$, for any $s_1, s_2 \in \mathbb{R}$ and any $s_3 \in [0,1]$, $s_4 \in [0,2]$*

$$\begin{aligned}
\|V-(U(y),0)\|_{H_x^{s_1}H_y^{s_2}} &\leq \|\omega_0\|_{H_x^{s_1-1}H_y^{s_2}},\\
\|V-(U(y),0)\|_{H_x^{s_1}H_y^{s_2}} &\leq \mathcal{O}(t^{-s_3})\|\omega_0\|_{H_x^{s_1-1}H_y^{s_2+s_3}},\\
\|V_2\|_{H_x^{s_1}H_y^{s_2}} &\leq \mathcal{O}(t^{-s_4})\|\omega_0\|_{H_x^{s_1-1}H_y^{s_2+s_4}}.
\end{aligned}$$

PROOF. We note that the change of variables $(k,\eta) \mapsto (k, \eta - kt)$, i.e. moving with the flow, is volume-preserving. For any L^2 estimate, instead of considering (8), we may thus equivalently work with

$$\begin{aligned}
\tilde{V}_1(t,k,\eta) &:= -\frac{i(\eta-kt)}{k^2+(\eta-kt)^2}\tilde{\omega}_0(k,\eta),\\
\tilde{V}_2(t,k,\eta) &:= \frac{ik}{k^2+(\eta-kt)^2}\tilde{\omega}_0(k,\eta).
\end{aligned} \tag{9}$$

These multipliers are uniformly bounded and converge to zero as $t \to \pm\infty$ for any $k \neq 0$.

In the case $k = 0$, the second multiplier is identically zero, while the first one yields

$$i\eta^{-1}\tilde{\omega}_0(0,\eta)$$

in Fourier space and

$$\partial_y^{-1}\langle\omega_0\rangle_x(y) =: U(y)$$

in real space. Subtracting $(U(y),0)$ from V, we may thus restrict to frequencies $k \neq 0$.

We note that the multipliers in (9) only tend to zero point-wise but not uniformly. More precisely, let $k \neq 0$ be given and consider some (very large) time $T > 0$. Then, for $\eta_0 := kT$, the time T is critical and hence, in this case,

$$\frac{ik}{k^2+(\eta_0-kT)^2} = \frac{ik}{k^2} = \frac{i}{k}.$$

As a consequence, for any given $k \neq 0$ and any $T > 0$, the estimate

$$\|\tilde{V}_1(T,k,\cdot)\|_{L^2(\mathbb{R})} \leq \frac{1}{|k|}\|\tilde{\omega}_0(k,\cdot)\|_{L^2(\mathbb{R})},$$

is sharp in the sense that there exists $\omega_0 \in L^2(\mathbb{T} \in \mathbb{R})$ (localized in Fourier space around (k,η_0) with $\eta_0 = kT$) such that

$$\|\tilde{V}_1(T,k,\cdot)\|_{L^2} \geq \frac{1}{2|k|}\|\tilde{\omega}_0(k,\cdot)\|_{L^2}. \tag{10}$$

We, however, stress that a given $\omega_0 \in L^2$ can not be concentrated in Fourier space around modes (k,kT) for arbitrarily many times T. In the following, we thus use the L^2 integrability of ω_0 and a cut-off in Fourier space to prove (I)-(III).

We claim that, for a given $\omega_0 \in L^2(\mathbb{T}\times\mathbb{R})$, for any $\epsilon > 0$ there exists a time $T = T(\omega_0,\epsilon) > 0$ such that, for any time $t > T$

$$\|v(t,x,y) - (U(y),0)\|_{L^2_{x,y}(\mathbb{T}\times\mathbb{R})} \leq \epsilon\|\omega_0\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{11}$$

The results (I)-(III) then follow from the claim by letting ϵ tend to zero.

Let thus $\omega_0 \in L^2(\mathbb{T}\times\mathbb{R})$ be given and let $\epsilon > 0$. Then, by the L^2 integrability of ω_0 and Plancherel's theorem, there exists an $R_1 = R_1(\omega_0,\epsilon)$, such that

$$\|\hat{\omega}_0(k,y)1_{|k|>R_1}\|_{L^2_{k,y}(\mathbb{Z}\times\mathbb{R})} \leq \frac{\epsilon}{4}\|\omega_0\|_{L^2(\mathbb{T}\times\mathbb{R})}.$$

As there are only finitely many $k \in \mathbb{Z}$ such that $|k| \leq R_1$, by the L^2 integrability in y, choosing $R_2 = R_2(\omega_0,\epsilon,R_1)$ sufficiently large, also

$$\|\tilde{\omega}_0(k,\eta)1_{|k|\leq R_1}1_{|\eta|>R_2}\|_{L^2_{k,\eta}(\mathbb{Z}\times\mathbb{R})} \leq \frac{\epsilon}{4}\|\omega_0\|_{L^2}.$$

Letting $R := \max(R_1,R_2)$, we hence obtain

$$\|\tilde{\omega}_0(k,\eta)(1 - 1_{|k|\leq R}1_{|\eta|\leq R})\|_{L^2_{k,\eta}(\mathbb{Z}\times\mathbb{R})} \leq \frac{\epsilon}{2}\|\omega_0\|_{L^2}. \tag{12}$$

Using this as a cut-off, we split

$$\omega_0 = \mathcal{F}^{-1}1_{|k|\leq R}1_{|\eta|\leq R}\mathcal{F}\omega_0 + \mathcal{F}^{-1}(1 - 1_{|k|\leq R}1_{|\eta|\leq R})\mathcal{F} =: \omega_I + \omega_{II}.$$

We note that ω_I is compactly supported in Fourier space and that ω_{II}, by (12), satisfies

$$\|\omega_{II}\|_{L^2} \leq \frac{\epsilon}{2}\|\omega_0\|_{L^2}.$$

Using (9), the contribution of ω_{II} to the velocity field V (and equivalently v) is controlled by

$$\left\|\frac{1}{\sqrt{k^2+(\eta-kt)^2}}\tilde{\omega}_{II}\right\|_{L^2_{k,\eta}((\mathbb{Z}\backslash\{0\})\times\mathbb{R})} \leq \|\omega_{II}\|_{L^2(\mathbb{T}\times\mathbb{R})} \leq \frac{\epsilon}{2}\|\omega_0\|_{L^2(\mathbb{T}\times\mathbb{R})}, \tag{13}$$

where we used (12) and that

$$\sup_{k\in(\mathbb{Z}\backslash\{0\}),\eta\in\mathbb{R}} \frac{1}{\sqrt{k^2+(\eta-kt)^2}} \leq 1.$$

In order to estimate the contribution by ω_I, we use that ω_I is compactly supported in Fourier space. Hence,

$$\left\|\frac{1}{\sqrt{k^2+(\eta-kt)^2}}\tilde{\omega}_I\right\|_{L^2_{k,\eta}((\mathbb{Z}\backslash\{0\})\times\mathbb{R})} \leq \|\omega_I\|_{L^2} \sup_{1\leq|k|\leq R,|\eta|\leq R} \frac{1}{\sqrt{k^2+(\eta-kt)^2}}.$$

Given R, there exists $T > 0$, such that for any $t > T$,

$$\sup_{1\leq|k|\leq R,|\eta|\leq R} \frac{1}{\sqrt{k^2+(\eta-kt)^2}} \leq \frac{\epsilon}{2}.$$

Therefore, for $t > T$,

(14)

$$\left\| \frac{1}{\sqrt{k^2+(\eta-kt)^2}} \tilde{\omega}_I(t,k,\eta) \right\|_{L^2_{k,\eta}((\mathbb{Z}\setminus\{0\})\times\mathbb{R})} \leq \frac{\epsilon}{2}\|\omega_I\|_{L^2(\mathbb{T}\times\mathbb{R})} \leq \frac{\epsilon}{2}\|\omega_0\|_{L^2(\mathbb{T}\times\mathbb{R})}.$$

Combining (13) and (14) then proves the claim, (11), and hence (I)-(III).

In order to prove (IV) and (V), we note that (9) can be improved by penalizing large frequencies η via regularity. For example, in the case of $\tilde{V}_1$, we estimate

$$\frac{i(\eta-kt)}{k^2+(\eta-kt)^2}\tilde{\omega}_0(k,\eta) = \frac{i(\eta-kt) < k >}{(k^2+(\eta-kt)^2) < \eta >} \left(\frac{< \eta >}{< k >} \tilde{\omega}_0(k,\eta) \right).$$

Controlling

$$\left| \frac{i(\eta-kt) < k >}{(k^2+(\eta-kt)^2) < \eta >} \right| = \mathcal{O}(t^{-1}),$$

then proves (IV). (V) is proven in the same way, using $\frac{<\eta>^2}{<\eta>^2}$ instead of $\frac{<\eta>}{<\eta>}$.

In order to prove (VI), we proceed as in estimate (10). Let thus $T > 0$ be a given (large) time and consider a vorticity ω_0, whose Fourier support is localized on

$$\{(\eta,k) : |\eta - kT| < \epsilon, |k| \approx 10\}$$

for some small ϵ. Then, considering (9) at time $t = T$, we see that on the support of ω_0 the multipliers are comparable to $\frac{1}{|k|}$ and hence

$$\|V|_{t=T} - (U(y),0)\|_{L^2} \approx \|\omega_0\|_{H_x^{-1}L_y^2}.$$

Recalling the support assumption on ω_0, a uniform estimate by $\mathcal{O}(t^{-1})$ can thus hold only if

$$< t >^1 \|\omega_0\|_{H_x^{-1}L_y^2} \approx \|\omega_0\|_{H_x^{-1}H_y^1} \lesssim 1.$$

In particular, control of just $\|\omega_0\|_{H_x^{-1}L_y^2}$ (or $\|\omega_0\|_{L_x^2H_y^s}, s < 1$) is not sufficient. The necessity of control of $\|\omega\|_{H_x^{-1}H_y^2}$ for (V) is proven in the same way.

The last statement, (VII), follows by noting that all preceding estimates are invariant under multiplication with $< k >^{s_1} < \eta >^{s_2}$ and interpolating between the cases $s_3 = 0,1$ and $s_4 = 0,2$, respectively. $\square$

Following the same approach as in Lemma 2.4 of the previous Section 2 (or by time-reversibility), we note that, prior to the respective critical times, the multipliers in (9) can be increasing:

Lemma 2.6. *Let η_0, k_0 with $k_0, \eta_0 < 0$ be given and let $\omega_0 \in L^2$ be localized in Fourier space around the frequency (k_0,η_0). Then the corresponding kinetic energy, $\|v\|_{L^2}^2$, is increasing for $0 < t < \frac{\eta_0}{k_0} =: t_c$.*

At the critical time t_c:

$$\|v(t_c)\|_{L^2} \geq \frac{1}{2}\frac{\sqrt{k_0^2+\eta_0^2}}{|k_0|}\|v(0)\|_{L^2}.$$

Proof. Using (9), we compute

$$\|v(t)\|_{L^2}^2 = \sum_k \int \frac{1}{k^2+(\eta-kt)^2}|\tilde{\omega}_0(k,\eta)|^2 d\eta.$$

Taking the localization of the support of $\tilde{\omega}_0$ into account, the multiplier is monotonically increasing until the critical time t_c. Evaluating at k_0, η_0 and at the times $0, t_c$, we thus obtain a quotient

$$\frac{1}{k_0^2 + (\eta_0 - t_c k_0)^2} \Big/ \frac{1}{k_0^2 + \eta_0^2} = \frac{k_0^2 + \eta_0^2}{k_0^2},$$

which yields the result. □

The growth and asymptotic decay, together with the involved regularity requirements, are known as the *Orr mechanism.*

As shown in [**Zil12**] and as we recall in Chapter 4, similar results also hold for fairly general monotone shear flows or even diffeomorphisms with some shearing structure. This also extends to more general domains, including the physically relevant finite periodic channel. In our proof, we follow a similar approach as given by Lin and Zeng, [**LZ11**].

4. Implications for Vlasov-Poisson

In Section 3 we have introduced the Vlasov-Poisson equations

$$\begin{aligned} \partial_t f + (v, F[f](t,x)) \cdot \nabla_{x,v} f = 0, \\ \nabla_{x,v} \cdot (v, F[f](t,x)) = 0, \\ F[f] = \nabla W *_x (\rho - \langle \rho \rangle_x), \\ \rho(t,x) = \int f(t,x,v) dv, \end{aligned} \tag{VP}$$

as well as their linearization around homogeneous densities $f_0(v)$

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h = -F[h] \cdot \nabla_v f_0, \\ F[h] = \nabla W *_x \rho - \langle \rho \rangle_x, \\ \rho(t,x) = \int h(t,x,v) dv. \end{aligned}$$

As discussed in Section 4, these equations exhibit neither dissipation, nor entropy increase or other usual damping mechanisms and indeed the Vlasov-Poisson equations are even time-reversible. It was thus a very surprising result due to Landau, [**Lan46**], that these equations exhibit damping behavior around homogeneous densities (on a linearized level).

In this section, we provide simplified heuristics for the damping mechanism by considering the free transport dynamics instead of the evolution by the linear and nonlinear Vlasov-Poisson equations. We thus *assume* the (small) force field to yield a negligible perturbation to the dynamics and treat f or h as *exact* solutions of free transport

$$\begin{aligned} \partial_t f + v \cdot \nabla f = 0, \\ (t,x,v) \in \mathbb{R} \times \mathbb{T} \times \mathbb{R}. \end{aligned} \tag{T}$$

and consider how the force field $F[f]$ evolves under the free transport dynamics. A discussion of actual linear Landau damping and Villani and Mouhot's results, [**MV11**], on nonlinear Landau damping follows in Chapter 3.

This section is based on a discussion in the works of Mouhot and Villani, [**MV11**], [**MV10a**], which was also used for the introductory discussion in the author's Master's thesis, [**Zil12**].

Considering some smooth initial data f_{in}, we recall that the explicit solution of the free transport equation is given by

$$f(t,x,v) = f_{in}(x - tv, v),$$
$$\tilde{f}(t,k,\eta) = \tilde{f}_{in}(k, \eta + kt),$$

and that f weakly converges to the x average. Thus,

$$\rho - <\rho>_x \rightharpoonup 0,$$

and

$$F[f] = \nabla W *_x (\rho - <\rho>_x) \to 0,$$

which is the observed damping phenomenon.

In order to obtain a more quantitative description, we note that

$$\mathcal{F}_x(F[f])(t,k) = k\hat{W}(k)\hat{\rho}(t,k) = k\hat{W}(k)\tilde{f}_{in}(k,kt).$$

LEMMA 2.7 (Damping under free transport dynamics). *Let f_{in} be given and let $f(t)$ be the solution of the free transport equations with initial datum f_0. Suppose further that for some $s > 0$, $\tilde{f}_{in}$ satisfies*

$$\sup_{\eta,k} | <\eta>^s \tilde{f}_{in}(k,\eta)| < C < \infty.$$

Then the density ρ associated to $f(t)$, i.e.

$$\rho(t,x) = \int f(t,x,v)dv,$$

satisfies

$$|\mathcal{F}\rho(t,k)| = |\tilde{f}_{in}(k,kt)| = \mathcal{O}(<kt>^{-s}).$$

Suppose further that f_{in} is Gevrey regular, i.e. that for all k and some $\lambda, \alpha > 0$, f_{in} satisfies

$$\|e^{\lambda|\eta|^\alpha} \tilde{f}_{in}(k,\eta)\|_{L^2_\eta} < C < \infty.$$

Then for any $\lambda' < \lambda$, ρ also satisfies

$$|\mathcal{F}\rho(t,k)| = |\tilde{f}_{in}(k,kt)| = \mathcal{O}(e^{-\lambda'|kt|^\alpha}).$$

Furthermore, $F[f]$ decays:

$$|\mathcal{F}_x(F[f])(t,k)| = |k\hat{W}(k)\tilde{f}_{in}(k,kt)| = \mathcal{O}(e^{-\lambda'|kt|^\alpha}).$$

PROOF. Considering $\eta = kt$ yields the results.

We remark that the supremum norm in Fourier space is less commonly used than for example L^2 based Sobolev norms:

$$\| <\eta>^s \tilde{f}_{in}(k,\eta)\|_{L^2_\eta}.$$

Results for those spaces can be obtained using embeddings. In the Gevrey case this distinction is less apparent, as, choosing λ' slightly smaller, one may freely use embeddings. $\square$

From this free transport model, we observe several heuristics:

- Higher regularity of f_{in} results in faster decay of ρ and $F[f]$.
- The regularity is necessary, i.e. uniform estimates on the decay rate require f_{in} to be regular.

- The dependence of the decaying factors on t is via kt. Hence, higher modes decay faster and a lower bound on $|k|$ is needed to obtain uniform decay rates.
- Gevrey 1 regularity, i.e. $\alpha = 1$, results in exponential decay of $F[f]$.

The exponential decay of the force field for the Vlasov-Poisson equations is the experimentally observed *Landau damping*.

We stress again that this section's discussion only provides *heuristics* for Landau damping, as we considered $F[f]$ for a solution f of free transport instead of $F[h]$ for a solution of the (linearized) Vlasov-Poisson equation. While it heuristically seems reasonable to expect that this behavior persists also for the (linearized) Vlasov-Poisson dynamics, provided the force field F is small initially, this, of course, has to be proven and assumptions on the potential W generating F have to be imposed.

In the following Chapter 3, we briefly sketch a proof of linear Landau damping and, in Section 2, we comment on the additional challenges and effects arising in the nonlinear setting such as plasma echoes and the resulting potential loss of regularity.

CHAPTER 3

Landau damping

In Section 4 we introduced the Vlasov-Poisson equations

$$
\begin{aligned}
\partial_t f + (v, F[f](t,x)) \cdot \nabla_{x,v} f &= 0, \\
\nabla_{x,v} \cdot (v, F[f](t,x)) &= 0, \\
F[f] &= \nabla W *_x (\rho - \langle \rho \rangle_x), \\
\rho(t,x) &= \int f(t,x,v) dv,
\end{aligned}
\tag{VP}
$$

as well as their linearization around homogeneous solutions, $f = f(v)$, and discussed similarities with the 2D Euler equations.

We have further seen that, under free transport dynamics, the force field is damped with a rate depending on the regularity of the initial data. As these heuristics neglect the changes of the dynamics by the force field, they only provide a motivation to expect Landau damping for small perturbations to homogeneous densities. It thus remains to establish that the actual solutions, evolving by the (linear) Vlasov-Poisson equations, behave similarly enough to free transport such that damping still holds.

In this chapter we briefly review the classical results on linear Landau damping and which type of control on F and the initial data is necessary to derive these results. Here a particular focus is on the technical approaches and the role of *gliding regularity* or moving with a chosen flow. Subsequently we review and discuss some of the central challenges arising for nonlinear Landau damping, such as loss of regularity and plasma echoes.

The discussions of both linear and nonlinear Landau damping follow the works of Villani and Mouhot, [**MV11**], [**MV10a**], [**MV10b**], which have also been the basis of the introductory discussions in [**Zil12**].

1. Linear damping

In this section, we consider the linearized Vlasov-Poisson equations around an analytic, homogeneous density $f_0(v)$

$$
\begin{aligned}
\partial_t h + v \cdot \nabla_x h &= -F[h] \cdot \nabla_v f_0, \\
F[h] &= \nabla W *_x (\rho - \langle \rho \rangle_x), \\
\rho(t,x) &= \int h(t,x,v) dv,
\end{aligned}
\tag{LVP}
$$

and sketch a proof of linear Landau damping for smooth solutions satisfying some of the common stability criteria.

As discussed in Chapter 2, we regard free transport as the main underlying dynamics. Hence, we employ Duhamel's formula to express the linear Vlasov-Poisson equation as an integral equation

$$\begin{aligned}
h(t,x,v) &= h_{in}(x-tv,v) - \int_0^t S(\tau, x-(t-\tau)v, v)d\tau, \\
F[h](t,x) &= \nabla W *_x (\rho - \langle \rho \rangle_x), \\
\rho(t,x) &= \int h(t,x,v)dv,
\end{aligned}$$

where

$$S(t,x,v) := F[h](t,x) \cdot \nabla_v f_0(v).$$

Employing a Fourier transform, we compute that h satisfies

$$\tilde{h}(t,k,\eta) = \tilde{h}_{in}(k, \eta + kt) - \int_0^t \tilde{S}(\tau, k, \eta + k(\tau - t))d\tau.$$

By the product structure of S

$$\tilde{S}(\tau,k,\eta) = 4\pi^2 k \cdot \eta \hat{W}(k) \hat{\rho}(\tau,k) \tilde{f}_0(\eta).$$

Considering $\eta = 0$, we thus obtain a closed equation for $\hat{\rho}(t,k) = \tilde{h}(t,k,0)$:

$$\begin{aligned}
\hat{\rho}(t,k) &= \tilde{h}_{in}(k,kt) - \int K_0(t-\tau,k)\hat{\rho}(\tau,k)d\tau, \\
K_0(t,k) :&= 4\pi^2 \hat{W}(k) \tilde{f}_0(kt)|k|^2 t.
\end{aligned} \tag{15}$$

As the force field, F, depends on h only via ρ, we in the following only consider $\eta = 0$.

We note that the integral in (15) is a convolution in time and that the equation is thus of the form of a *Volterra equation*, i.e.

$$\phi(t) = a(t) + \int_0^t K(t-\tau)\phi(\tau)d\tau, \tag{Volterra}$$

with k being an additional parameter. Introducing the complex *Laplace transform*, (c.f. [**MV10a**, section 3.3.]):

$$\phi^L(\xi) := \int_0^\infty e^{2\pi \bar{\xi} t} \phi(t)dt,$$

for $\xi \in \mathbb{C}$, which we for the moment suppose to exist for $\xi \in \mathbb{R}$, the convolution is thus transformed into a product. Hence,

$$\phi^L = a^L + K^L \phi^L \Leftrightarrow \phi^L = \frac{a^L}{1-K^L}.$$

Supposing a to decay exponentially fast and K^L to be bounded away from one uniformly, the following lemma shows that $\phi(t)$ decays exponentially fast as well.

Lemma 3.1 ([**MV10a**, page 36]). *Let $K : \mathbb{R}_+ \to \mathbb{C}$ be such that*

$$\begin{aligned}
|K(t)| &\lesssim e^{-\lambda_0 t}, \\
|K^L(\xi) - 1| &\geq \kappa > 0 \text{ for } 0 \leq \Re(\xi) \leq \Lambda.
\end{aligned}$$

Suppose further that $|a(t)| \lesssim e^{-\lambda t}$. Then the solution ϕ of

$$\phi(t) = a(t) + \int_0^t K(t-\tau)\phi(\tau)d\tau$$

exists and for any $0 \leq \lambda' < \min(\lambda, \lambda_0)$,

$$|\phi(t)| \lesssim e^{-\lambda' t}.$$

Corollary 3.1. *Let h be a solution of the linearized Vlasov-Poisson equation and suppose that for all k*

$$a(t) = \tilde{h}_{in}(k, kt),$$
$$K(t) = 4\pi^2 \hat{W}(k) \tilde{f}_0(kt)|k|^2 t,$$

satisfy the assumptions of the Lemma 3.1.

Then

$$F[h] = \nabla W *_x (\rho - \langle \rho \rangle_x)$$

decays exponentially fast.

This decay of the force field is measured in physical experiments and is commonly referred to as *Landau damping.*

As the assumptions of the corollary are difficult to verify directly, in the following we discuss weaker, sufficient conditions. The condition $|a(t)| \lesssim e^{-\lambda t}$ requires exponential decay of $\tilde{h}$ with respect to η, i.e.

$$\sup_k |\tilde{h}_{in}(k, \eta)| = \mathcal{O}(e^{-\lambda|\eta|}).$$

For this it is sufficient to assume that h_{in} is in a suitable Gevrey class:

Definition 3.1 (Gevrey). A function $u : \mathbb{R} \mapsto \mathbb{C}$ is in the Gevrey class $\mathcal{G}_s, s > 0$, iff for some $\lambda > 0$,

$$\| e^{\lambda |\eta|^{\frac{1}{s}}} (\mathcal{F}u)(\eta) \|_{L^2_\eta(\mathbb{R})} < \infty.$$

In particular, smaller s yield a stronger condition and $s = 1$ corresponds to analytic regularity.

For the exponential decay of the kernel, $|K(t)| \lesssim e^{-\lambda t}$, by a similar argument it is sufficient to require that f_0 is in a suitable Gevrey class.

It remains to obtain a sufficient condition for the bound of the kernel:

$$|K^L(\xi) - 1| \geq \kappa > 0 \text{ for } 0 \leq \Re(\xi) \leq \Lambda. \tag{16}$$

For this purpose, we use that

$$|K^L(\xi) - 1| \geq \min(\Im(K^L)(\xi), |1 - |K^L(\xi)||, |1 - \Re(K^L(\xi))|).$$

Hence, it suffices to verify that either:

(A) $|\Im(K^L(\xi))| > 0$.

(B) $|\Re(K^L(\xi))| < 1$ or $|K^L(\xi)| < 1$.

It can be shown ([**MV10a**], section 4), that for $\lambda \in \mathbb{R}, w > 0$, K^L satisfies:

$$K^L((\lambda - iw)|k|, |k|) = \hat{W}(k) \int_{\mathbb{R}} \frac{f_0(v)}{v - w + i\lambda} dv. \tag{17}$$

We note that

$$\left|\frac{1}{v - w + i\lambda}\right| \leq \frac{1}{|\lambda|}$$

and similarly

$$\left|\frac{1}{v - w + i\lambda}\right| \lesssim \frac{1}{|w|}$$

for $|v| \leq |w|/2$. Letting λ or w tend to infinity and using the decay of f_0 at infinity, $K^L((\lambda - iw)|k|, |k|)$ tends to zero and hence (B) is satisfied.

It therefore remains to verify (A) and (B) on the compact set

$$\{|w| \leq \Omega, 0 \leq \lambda \leq \Lambda\}$$

for given $\Omega, \Lambda > 0$. Considering the limit $\lambda \downarrow 0$ in (17), the Plemelj formula yields

$$\begin{aligned}
&\lim_{\lambda \downarrow 0} K^L((w - i\lambda)|k|, |k|) \\
&= \hat{W}(k) \left(p.v. \int \frac{f_0'(v)}{v - w} dv - i\pi f_0'(w) \right) =: z(k, w),
\end{aligned}$$

where $p.v.$ denotes the principal value.

Considering (A) at such points and letting Λ be sufficiently small in (16), by compactness of $|w| \leq \Omega$ we may further restrict to (neighborhoods of) w such that the imaginary part vanishes. As $\hat{W}$ is real-valued, by the product structure of (17), either $f_0'(w) = 0$ or $\hat{W}(k) = 0$. In the latter case however, the real part vanishes as well and hence (B) is satisfied.

The control of points $w \in \mathbb{R}$ such that $f_0'(w) = 0$ is formalized in the Penrose stability criterion:

Definition 3.2 (Penrose stability criterion [**MV10a**]). Let $\hat{W}$ be given. Then f_0 satisfies the Penrose stability criterion, if for some $\kappa > 0$ and for all $w \in \mathbb{R}$

$$f_0'(w) = 0 \Rightarrow \hat{W}(k) p.v. \int \frac{f_0'(v)}{v - w} dv < 1 - \kappa,$$

where again $p.v.$ denotes the principal value.

By the preceding discussion, linear Landau damping thus holds, if

- h_{in} is in a suitable Gevrey class.
- f_0 is in a suitable Gevrey class.
- f_0 satisfies the Penrose stability criterion.

Mouhot and Villani, [**MV10a**], discuss the application of this criterion to both gravitational and electric interaction, which we briefly reproduce in the following.

Let f_0 be a Gaussian density with standard deviation β and W corresponding to gravitational interaction, i.e.

$$\begin{aligned}
\hat{W}(k) &= \frac{-\mathcal{G}}{\pi |k|^2}, \\
f_0(v) &= \rho_0 \sqrt{\frac{\beta}{2\pi}} e^{-\beta v^2}.
\end{aligned}$$

The Penrose stability criterion then reads

$$1 > \frac{\mathcal{G} \rho_0 \beta}{\pi |k|^2}.$$

If we consider periodic perturbations with period length L, then $k \in \frac{\pi}{L}\mathbb{Z} \setminus \{0\}$. Taking the supremum over such k, the stability criterion requires an upper bound on L:

$$L < \sqrt{\frac{\pi}{\mathcal{G} \rho_0 \beta}} =: L_J.$$

L_J is called the *Jeans length*.

When considering Coulomb interaction, the sign is changed and

$$1 > 0 > -\frac{\mathcal{Q} \rho_0 \beta}{\pi |k|^2},$$

for all k. The stability criterion is thus always satisfied.

2. Nonlinear damping

In the previous section, we have seen that Landau damping holds on a linearized level. A natural question is thus whether this behavior persists for the nonlinear dynamics.

Here, it has been noted by Backus, [**Bac60**], that the linear dynamics can a priori not be expected to stay close to the nonlinear dynamics for large times. More precisely, suppose that the nonlinearity is negligible and that thus h approximately evolves by free transport. Then Sobolev norms of h asymptotically grow with algebraic rates, e.g.

$$\mathcal{F}(\nabla_v h) \approx \eta \tilde{h}_{in}(k, \eta + kt)$$

grows linearly in time, which contradicts the smallness assumption of the linearisation for large times (c.f. [**MV10a**, chapter 4.3]). Thus, the asymptotic behavior of the linear equation can a priori not be expected to provide information on the asymptotic behavior of the nonlinear equation.

A further indication that the nonlinear dynamics could be very different from the linear dynamics, is the phenomenon of so-called plasma echoes, experimentally observed in 1968, [**MWGO68**]. In the experiment two modes are excited at different times and damping is observed. However, after some amount of time, which can be computed from the modes, a peak is observed. The damping has thus not removed information but rather transfered it to high frequencies, where both excitations can interact to strongly influence a third mode, which at its critical time manifests as a peak.

Despite these obstructions Villani and Mouhot in their seminal work, [**MV11**], proved nonlinear Landau damping for Gevrey regular perturbations around analytic, homogeneous profiles. Following this work and using similar but slightly different methods, Bedrossian and Masmoudi, [**BM13b**], proved nonlinear inviscid damping for small Gevrey regular perturbations to Couette flow. These methods have subsequently also been adapted to Landau damping, [**BMM13**].

The main result of [**MV11**] is given by the following theorem.

Definition 3.3 (Damping Condition (L)). We say that f_0 and W satisfy the condition **(L)**, if there are constants $C_0, \lambda, \kappa > 0$ such that for any $\eta \in \mathbb{R}^d$, $|\tilde{f}^0(\eta)| \leq C_0 e^{-2\pi\lambda|\eta|}$; and for any $\xi \in \mathbb{C}$ with $0 \leq \Re\xi < \lambda$,

$$\inf_{k \in \mathbb{Z}^d} |K^L(\xi, k) - 1| \geq \kappa. \tag{L}$$

Theorem 3.1 (Nonlinear Landau damping [**MV11**, Theorem 2.6]). *Let $f^0 : \mathbb{R}^d \to \mathbb{R}_+$ be an analytic velocity profile. Let $L > 0$ and $W : \mathbb{T}_L^d \to \mathbb{R}$ be an interaction potential satisfying*

$$\forall k \in \mathbb{Z}^d, \qquad |\hat{W}^L(k)| \leq \frac{C_W}{|k|^{1+\gamma}}$$

*for some constants $C_W > 0, \eta \geq 1$. Assume that f^0 and W satisfy the stability condition **(L)** with some constants $\lambda, \kappa > 0$; further assume that, for the same parameter,*

$$\sup_{\eta \in \mathbb{R}^d} \left(|\tilde{f}^0(\eta)| e^{2\pi|\eta|} \right) \leq C_0,$$

$$\sum_{n \in \mathbb{N}_0^d} \frac{\lambda^n}{n!} \|\nabla_v^n f^0\|_{L^1}(\mathbb{R}^d) \leq C_0 < +\infty.$$

Then for any $0 < \lambda' < \lambda$, $\beta > 0$, $0 < \mu' < \mu$, *there is* $\epsilon = \epsilon(d, L, C_W, C_0)$, $\kappa, \lambda, \lambda', \mu, \mu', \beta, \gamma)$ *with the following property: if* $f_i = f_i(x, v)$ *is an initial datum satisfying*

$$\delta := \|f_i - f^0\|_{\lambda,\mu} + \iint_{\mathbb{T}^d_L \times \mathbb{R}^d} |f_i - f^0| e^{\beta|v|} dv dx \le \epsilon$$

then

- *the unique classical solution* f *to the nonlinear Vlasov equation*

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - (\nabla W * \rho) \cdot \nabla_v f = 0, \qquad \rho = \int_{\mathbb{R}^d} f dv, \tag{Vlasov}$$

  with initial datum $f(0, \cdot) = f_i$, *converges in the strong topology as* $t \to \pm\infty$, *with rate* $\mathcal{O}(e^{-2\pi\lambda'|t|})$, *to a spatially homogeneous equilibrium* $f_{\pm\infty}$;
- *the density* $\rho(t, x) = \int f(t, x, v) dv$ *converges in the strong topology as* $t \to \pm\infty$, *with rate* $\mathcal{O}(e^{-2\pi\lambda'|t|})$, *to the constant density*

$$\rho_\infty = \frac{1}{L^d} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d)_L} f_i(x, v) dx dv;$$

  in particular the force $F = -\nabla W * \rho$ *converges exponentially fast to* 0.
- *the spatial average* $\langle f \rangle(t, v) = \int f(t, x, v) dx$ *converges in the strong topology as* $t \to \pm\infty$, *with rate* $\mathcal{O}(e^{-2\pi\lambda'|t|})$, *to* $f_{\pm\infty}$. *More precisely, there are* $C > 0$, *and spatially homogeneous distributions* $f_{+\infty}(v)$ *and* $f_{-\infty}(v)$, *depending continuously on* f_i *and* W, *such that*

$$\sup_{t \in \mathbb{R}} \|f(t, x + vt, v) - f^0(v)\|_{\lambda',\mu'} \le C\delta;$$

$$\forall \eta \in \mathbb{R}^d, \qquad |\tilde{f}_{\pm\infty} - \tilde{f}^0(\eta)| \le C\delta e^{-2\pi\lambda'|\eta|}$$

  and

$$\forall (k, \eta) \in \mathbb{Z}^d \times \mathbb{R}^d, \qquad \left|L^{-d} \tilde{f}^L(t, k, \eta) - \tilde{f}_{+\infty}(\eta) 1_{k=0}\right| = \mathcal{O}(e^{-2\pi \frac{\lambda'}{L} t}) \quad \text{as } t \to +\infty;$$

$$\forall (k, \eta) \in \mathbb{Z}^d \times \mathbb{R}^d, \qquad \left|L^{-d} \tilde{f}^L(t, k, \eta) - \tilde{f}_{-\infty}(\eta) 1_{k=0}\right| = \mathcal{O}(e^{-2\pi \frac{\lambda'}{L} |t|}) \quad \text{as } t \to -\infty;$$

$$\forall r \in \mathbb{N}, \qquad \|\rho(t, \cdot) - \rho_\infty\|_{C^r(\mathbb{T}^d)} = \mathcal{O}(e^{-2\pi \frac{\lambda'}{L} |t|}) \qquad \text{as } |t| \to \infty;$$

$$\forall r \in \mathbb{N}, \qquad \|F(t, \cdot)\|_{C^r(\mathbb{T}^d)} = \mathcal{O}(e^{-2\pi \frac{\lambda'}{L} |t|}) \qquad \text{as } |t| \to \infty;$$

$$\forall r \in \mathbb{N}, \forall \sigma > 0, \qquad \|F(t, \cdot)\|_{C^r_\sigma(\mathbb{R}^d_v)} = \mathcal{O}((e^{-2\pi \frac{\lambda'}{L} |t|}) \qquad \text{as } |t| \to \infty.$$

In this statement C^r *stands for the usual norm on* r *times continuously differentiable functions, and* C^r_σ *involves in addition moments of order* σ, *namely* $\|f\|_{C^r_\sigma} = \sup_{r' \le r, v \in \mathbb{R}^d} |f^{(r')}(v)(1 + |v|^\sigma)|$.

Villani also gives some sufficient conditions for (L) to hold (cf. [**MV11**, Proposition 2.1]).

2.1. Techniques for the nonlinear setting. In this subsection we briefly comment on some of the main techniques used in the proof of nonlinear Landau damping. For a more in-depth discussion we refer to Villani's various publications [**MV10a**], [**MV10b**], [**MV11**] as well as [**BMM13**] and [**BM13b**].

As we have seen in the previous section, in the linearized setting the equation decouples in frequency and a fairly direct approach via explicit solutions. Therefore, the integral representation using Duhamel's formula is possible. The additional coupling in the nonlinear problem removes this simplifying structure and thus a more abstract approach is necessary.

We briefly comment on the main tools used:

- As in the linearized setting, regularity can only be expected when moving with the flow. This is formalized as *gliding regularity*, that is norms having the transport frozen in as a parameter:
$$\| \cdot \|_{X_t} := \| e^{-tS} \cdot \|_X,$$
where $\| \cdot \|_X$ is a given norm and e^{tS} denotes the solution operator of the underlying transport.
- h exhibits *self-improvement*, i.e. if F improves (regularity, decay) then so does h.
- On short time scales h improves and thus also F.
- As compositions play an important role, many different norms adapted to the problem are introduced and their behavior under composition is analyzed.
- A Newton scheme together with the fast decay (analytic $\rightarrow$ exponential decay) and estimates for the quasi-linear equation are used to deduce the result.

In the following Chapters 4 and 5, we in particular make use of gliding regularity and spaces adapted to the problem in order to prove linear inviscid damping for general regular, strictly monotone shear flows. There we also show that boundary effects in a finite periodic channel $\mathbb{T} \times [0, 1]$ have a large effect on the dynamics. As the Vlasov-Poisson equations are posed on the phase space $\mathbb{T} \times \mathbb{R}$, there is no corresponding setting for Landau damping.

Subsequently, in Chapter 6, we briefly comment on the additional challenges in the nonlinear setting as seen in Bedrossian and Masmoudi's work, [**BM13a**], on nonlinear inviscid damping for Couette flow.

CHAPTER 4

Linear inviscid damping for monotone shear flows

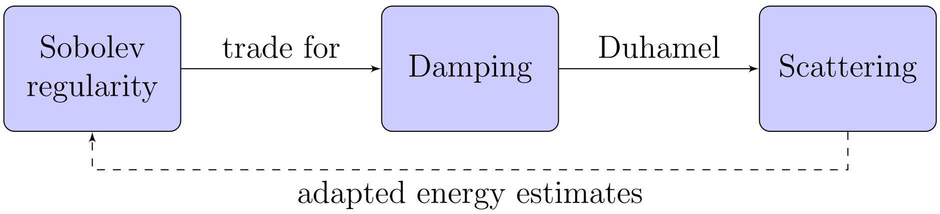
As we have discussed in Section 3 of Chapter 2, the linearized Euler equations around Couette flow in an infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, exhibit linear inviscid damping. That is, any perturbation, $(v, \omega) \in L^2(\mathbb{T} \times \mathbb{R}) \times H^2(\mathbb{T} \times \mathbb{R})$, is damped to a shear flow with algebraic rates:

$$\begin{aligned} \|v_1 - \langle v_1 \rangle_x\|_{L^2} &\leq \mathcal{O}(t^{-1}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H_x^{-1} H_y^1}, \\ \|v_2\|_{L^2} &\leq \mathcal{O}(t^{-2}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H_x^{-1} H_y^2}, \end{aligned} \tag{18}$$

and these decay rates are optimal. Going beyond this explicitly solvable (and in this sense trivial) setting, however has remained open until recently.

As the main result of this thesis, in this chapter, we, for the first time, rigorously prove linear inviscid damping for a large class of general, monotone shear flows. Furthermore, in addition to the common setting of an infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, we also prove linear inviscid damping in the physically relevant setting of a finite periodic channel, $\mathbb{T} \times [0, 1]$, with impermeable walls. There, we show that the latter setting is not only technically more challenging due to the lack of Fourier methods and similar tools, but that boundary effects play a non-negligible role.

Our strategy to prove linear inviscid damping is summarized by the following figure:



Considering the linearized Euler equations around a strictly monotone shear flow $U(y)$,

$$\partial_t \omega + U(y) \partial_x \omega = U'' v_2,$$

as a perturbation around the underlying transport by the shear, we introduce the *scattered vorticity*

$$W(t, x, y) := \omega(t, x - tU(y), y).$$

As a first step and as we recall and expand on in Section 1, in the author's Master's thesis, [**Zil12**], it has been shown that, *assuming* regularity of W, i.e. that

$$\|W(t)\|_{L_x^2 H_y^2} < C < \infty$$

for all $t \geq 0$, damping estimates of the form (18) can be extended to a class of general, strictly monotone shear flows.

THEOREM 4.1 (Damping). *Let $U(y)$ be a strictly monotone, regular shear flow, i.e. $U' > c > 0$ and $U' \in W^{2,\infty}$. Then for any solution ω of the linearized 2D Euler equations in either the infinite periodic channel or the finite periodic channel, denoting*

$$W(t,x,y) := \omega(t, x - tU(y), y) - \langle \omega_0 \rangle_x(y),$$

the perturbation to the velocity field is controlled by

$$\begin{aligned} \|v_1(t) - \langle v_1 \rangle_x\|_{L^2} &\leq \mathcal{O}(t^{-1}) \|W(t)\|_{H_x^{-1} H_y^1}, \\ \|v_2(t)\|_{L^2} &\leq \mathcal{O}(t^{-2}) \|W(t)\|_{H_x^{-1} H_y^2}. \end{aligned}$$

Assuming control of $\|W\|_{L_x^2 H_y^2}$ uniformly in time, the velocity perturbation hence decays with the optimal algebraic rates. As a consequence, under the same *assumption*, it can be shown that ω converges to a free solution of the underlying transport equation, i.e. that W converges to some asymptotic profile.

THEOREM 4.2 (Scattering). *Let W be a solution of the linearized 2D Euler equations in either the infinite periodic channel or finite periodic channel and suppose that $U'' \in L^\infty$ and that*

$$\|v_2(t)\|_{L^2} = \mathcal{O}(t^{-1-\epsilon})$$

for some $\epsilon > 0$. Then there exists $W_\infty \in L^2$, such that

$$W(t) \xrightarrow{L^2} W_\infty,$$

as $t \to \infty$.

It is thus shown that linear inviscid damping, like Landau damping, is at its core a *problem of regularity* and stability. We further stress that the damping result necessarily costs regularity, as the underlying transport is a unitary operation and can therefore not yield any decay. Our strategy, which is depicted in the figure, can thus not be easily closed. Instead, in order to prove stability results and in particular a control of the regularity of $W(t)$,

$$\|W(t)\|_{L_x^2 H_y^2} \lesssim \|\omega_0\|_{L_x^2 H_y^2},$$

we thus have to invest considerable technical effort to make use of finer properties of the dynamics.
The main results of this Chapter are given by Theorem 4.11 in Section 2 and Theorem 4.15 in Section 3, which establish $L_x^2 H_y^2$ stability – and thus linear inviscid damping with optimal algebraic rates as well as scattering– for the infinite periodic channel and finite periodic channel, respectively.

More precisely, Theorem 4.11 establishes stability in arbitrary Sobolev spaces $H_x^m H^s(\mathbb{T} \times \mathbb{R}), j, m \in \mathbb{N}_0$. In contrast, in the setting of a finite channel, we establish stability in $H_x^m H_y^1(\mathbb{T} \times [0,1]), m \in \mathbb{N}_0$, for general perturbations. However, due to boundary effects, for stability in $H_x^m H_y^2(\mathbb{T} \times [0,1]), m \in \mathbb{N}_0$, we have to additionally require that our perturbations ω_0 have vanishing zero Dirichlet data, $\omega_0|_{y=0,1} = 0$. In section 6, we show for the setting of Couette flow that this not only a technical restriction but that otherwise $\partial_y W|_{y=0,1}$ asymptotically blows up with a logarithmic rate and in particular forbids high regularity.

In Chapter 5, we further study these boundary effects in more detail and for general shear flows and show that, depending on the perturbations considered, the fractional Sobolev spaces $H_x^m H_y^{3/2}$ and $H_x^m H_y^{5/2}$ are critical for stability results. More precisely, we show stability in all subcritical (periodic, for technical reasons) fractional Sobolev spaces and blow-up in the supercritical spaces.

Theorem 4.16 combines the stability, damping and scattering result and thus proves linear inviscid damping for a large class of monotone shear flows, both in an infinite periodic channel, as well as in a finite period channel:

THEOREM 4.3. *Let Ω be either the infinite periodic channel of period L, $\mathbb{T}_L \times \mathbb{R}$, or the finite periodic channel, $\mathbb{T}_L \times [0,1]$, with impermeable walls. Let $U : \mathbb{R} \mapsto \mathbb{R}$ be a monotone shear flow and suppose that there exists $c > 0$ such that*

$$0 < c < U' < c^{-1} < \infty,$$

and that $U''(U^{-1}(\cdot)) \in W^{3,\infty}$. Suppose further that

$$L \| U''(U^{-1}(\cdot)) \|_{W^{j+1,\infty}}$$

is sufficiently small.

Then, for any $\omega_0 \in L^2_x H^2_y(\mathbb{T}_L \times \mathbb{R})$ or $\omega_0 \in L^2_x H^2_y(\mathbb{T}_L \times \mathbb{R})$ with zero Dirichlet data, $\omega_0|_{y=0,1} = 0$, there exists a function $W_\infty \in L^2_x H^2_y(\Omega)$, such that the solution ω of the linearized Euler equations on Ω with initial data ω_0, satisfies

(Stability) $$\|W(t)\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y},$$

(Damping) $$\|v(t) - \langle v \rangle_x\|_{L^2} = \mathcal{O}(t^{-1}) \|\omega_0\|_{L^2_x H^2_y},$$
$$\|v_2(t)\|_{L^2} = \mathcal{O}(t^{-2}) \|\omega_0\|_{L^2_x H^2_y},$$

(Scattering) $$W(t) \to_{L^2} W_\infty, \text{ as } t \to \infty,$$

where

$$W(t,x,y) = \omega(t, x - tU(y), y).$$

1. Damping under regularity assumptions

In the following, we extend the damping result for Couette flow of Section 3 of Chapter 2 to more general shear flows $(U(y), 0)$. Here, we consider the settings of an infinite channel of period L, $\mathbb{T}_L \times \mathbb{R}$, as well as of a finite periodic channel, $\mathbb{T}_L \times [0,1]$. In both settings the linearized Euler equations around a shear flow $(U(y), 0)$ are given by:

$$\begin{aligned} \partial_t \omega + U(y) \partial_x \omega &= U'' v_2, \\ v_2 &= \partial_x \phi, \\ \Delta \phi &= \omega, \end{aligned} \tag{19}$$

where for the infinite channel, the velocity field $v = \nabla^\perp \phi$ is required to be integrable, i.e.

$$\phi \in \dot{H}^1(\mathbb{T}_L \times \mathbb{R}),$$

and in the case of a finite channel, $\mathbb{T}_L \times [0,1]$, we consider impermeable walls, i.e. we additionally

$$v_2 = 0 \text{ for } y \in \{0, 1\}.$$

For simplicity of notation, in the following we write $\mathbb{T}$ to denote the torus of period 1, $\mathbb{T}_1$, if there is no danger of confusion.

In view of the damping results of Chapter 2, we consider the right-hand-side, $U'' v_2$, to be a perturbation and introduce the *scattered vorticity*

$$W(t,x,y) := \omega(t, x - tU(y), y). \tag{20}$$

As for Couette flow, taking the x average of the equation, we see that

$$\langle W \rangle_x (t,y) = \langle \omega \rangle_x (t,y) = \langle \omega_0 \rangle_x (y) \tag{21}$$

is independent of time. By linearity and writing

$$\omega_0 = (\omega_0 - \langle \omega_0 \rangle_x) + \langle \omega_0 \rangle_x,$$

in the following without loss of generality we only consider the case $\langle W \rangle_x \equiv 0$.

The results of Section 3 of Chapter 2 for Couette flow show that regularity of W is needed to establish damping results for the velocity field. In this section, we *assume* W to be of regularity comparable to ω_0 also in high Sobolev norms, uniformly in time.

The proof of stability of W and hence control of

$$\|W(t)\|_{L^2_x H^2_y},$$

which is the main result of this chapter, is obtained in Sections 2 and 3.

Using the regularity, we establish damping results with the same optimal algebraic rates as for Couette flow also for general, strictly monotone shear flows, where the bounds are now in terms of W instead of ω_0. In Section 1.1, these results are further generalized and reformulated in terms of the respective flow maps.

As we consider general shear flows and also the setting of a finite periodic channel, Fourier methods are not available anymore. We therefore obtain results by duality in analogy to classical stationary phase arguments and as an extension of [**LZ11**] and [**BM10**, Appendix A.1].

THEOREM 4.4 (Generalization of [**LZ11**, Theorem 3]; [**Zil12**]). *Let Ω be either the infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$, or the finite periodic channel, $\mathbb{T}_L \times [0,1]$. Let ω be a solution to the linearized Euler equations,* (19)*, around a strictly monotone shear flow $U(y)$, on the domain Ω. Suppose further that the initial datum, ω_0, satisfies $\langle \omega_0 \rangle_x = 0$ and that $\frac{1}{U'} \in W^{2,\infty}(\Omega)$. Then the following statements hold:*

(1) If $W(t) \in H^{-1}_x H^1_y(\Omega)$ for all times, then

$$\|v(t) - \langle v \rangle_x\|_{L^2(\Omega)} = \mathcal{O}(t^{-1}) \|W(t)\|_{H^{-1}_x H^1_y(\Omega)}, \text{ as } t \to \pm\infty.$$

(2) If $W(t) \in H^{-1}_x H^2_y(\Omega)$ for all times, then

$$\|v_2(t)\|_{L^2(\Omega)} = \mathcal{O}(t^{-2}) \|W(t)\|_{H^{-1}_x H^2_y(\Omega)}, \text{ as } t \to \pm\infty.$$

PROOF. The results are established by testing. More precisely, in the infinite channel case, denoting the stream function by ϕ, v satisfies

$$\begin{aligned} \|v - \langle v \rangle_x\|^2_{L^2(\mathbb{T}_L \times \mathbb{R})} \leq \|v\|^2_{L^2} &= \iint_{\mathbb{T}_L \times \mathbb{R}} |\nabla^\perp \phi|^2 = \iint_{\mathbb{T}_L \times \mathbb{R}} |\nabla \phi|^2 \\ &= -\iint_{\mathbb{T}_L \times \mathbb{R}} \phi \Delta \phi = -\iint_{\mathbb{T}_L \times \mathbb{R}} \phi \omega, \end{aligned} \tag{22}$$

where we used that ϕ decays sufficiently rapidly for $|y| \to \infty$ and that $\Delta \phi = \omega$. Hence,

$$\|v - \langle v \rangle_x\|_{L^2(\mathbb{T}_L \times \mathbb{R})} \lesssim \sup_{\psi \in H^1(\mathbb{T}_L \times \mathbb{R}), \|\psi\|_{H^1} \leq 1} \iint_{\mathbb{T}_L \times \mathbb{R}} \psi \omega. \tag{23}$$

It can be shown (see [**Lin04**, Lemma 3]), that an estimate of this form also holds in the setting of a finite channel, where the supremum is instead taken over elements of $\hat{H}^1 := \{\psi \in H^1(\mathbb{T}_L \times [0,1]) : \psi = 0 \text{ for } y \in \{0,1\}\}$, i.e.

$$\|v\|_{L^2(\mathbb{T}_L \times [0,1])} \lesssim \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \iint_{\mathbb{T}_L \times [0,1]} \psi \omega. \tag{24}$$

Indeed, let ϕ be the stream function corresponding to v, then

$$\nabla^{\perp}(\phi - \langle \phi \rangle_x) = v - \langle v \rangle_x,$$
$$\phi - \langle \phi \rangle_x|_{y=0,1} = 0,$$

where we used that on the boundary, $y \in \{0,1\}$

$$0 = v_2 = \partial_x \phi,$$

and hence $\phi - \langle \phi \rangle_x|_{y=0,1} = 0$. An integration by parts as in (22) thus yields no boundary contributions and hence the same estimate.

For simplicity of notation, in the following we use $\hat{H}^1$ to also denote $H^1(\mathbb{T}_L \times \mathbb{R})$, so that both (23) and (24) read the same.

We further introduce $f_k(t,y) := \mathcal{F}_x W(t,k,y)$. Then,

$$\begin{aligned}\|v - \langle v \rangle_x\|_{L^2(\Omega)} &\lesssim \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \left| \iint_\Omega \psi \omega \right| \\ &= \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \left| \sum_{k \neq 0} \int \psi_{-k} f_k e^{iktU(y)} \right|.\end{aligned}$$

We integrate by parts to obtain

$$\int \psi_{-k} f_k e^{iktU(y)} dy = - \int \frac{e^{iktU(y)}}{ikt} \partial_y \left(\frac{\psi_{-k} f_k}{U'} \right) dy, \tag{25}$$

where, in the case of a finite channel, the boundary terms

$$\left. \frac{e^{iktU(y)}}{iktU'(y)} \psi_{-k} f_k \right|_{y=0}^{1}$$

vanish as ψ vanishes on the boundary. Using the strict monotonicity of U and Hölder's inequality, we thus bound

$$\|v(t) - \langle v \rangle_x\|_{L^2(\Omega)} \lesssim \sup_{\psi \in \hat{H}^1, \|\psi\|_{H^1} \leq 1} \mathcal{O}(t^{-1}) \|W(t)\|_{H_x^{-1} H_y^1} \|\psi\|_{H^1}, \tag{26}$$

which establishes the first statement.

In order to bound v_2, we proceed slightly differently. Note that v_2 satisfies

$$\Delta v_2 = \partial_x \omega. \tag{27}$$

We thus introduce a potential ψ such that

$$\Delta \psi = v_2.$$

In the case of an infinite channel, we require that $\nabla \psi \in L^2(\mathbb{T}_L \times \mathbb{R})$. For the finite channel, we additionally require zero Dirichlet conditions, i.e.

$$\psi = 0, \text{ for } y \in \{0,1\}. \tag{28}$$

Therefore,

$$\begin{aligned}\iint_{\mathbb{T}_L \times [0,1]} \partial_x \omega \psi &= \iint_{\mathbb{T}_L \times [0,1]} \Delta v_2 \psi \\ &= \int_0^1 \psi \partial_x v_2|_{x=0}^L \, dy + \int_{\mathbb{T}_L} \psi \partial_y v_2|_{y=0}^1 \, dx - \iint_{\mathbb{T}_L \times [0,1]} \nabla v_2 \cdot \nabla \psi \\ &= - \int_0^1 v_2 \partial_x \psi|_{x=0}^L \, dy - \int_{\mathbb{T}_L} v_2 \partial_y \psi|_{y=0}^1 \, dx + \iint v_2 \Delta \psi = \|v_2\|^2_{L^2(\mathbb{T}_L \times [0,1])},\end{aligned}$$

where we used periodicity in x and that v_2 and ψ vanish whenever $y \in \{0,1\}$. Hence, for both the infinite and finite channel,

$$\|v\|^2_{L^2(\Omega)} = \iint_\Omega \partial_x \omega \psi. \tag{29}$$

Using (29), we compute

$$\begin{aligned}\|v_2\|^2_{L^2(\Omega)} &= \iint \partial_x \omega \psi = \sum_k \int ike^{iktU(y)} f_k \psi_{-k} \\ &= \sum_k \int \frac{e^{iktU(y)}}{t} \partial_y \left(\frac{f_k \psi_{-k}}{U'} \right).\end{aligned}$$

Integrating by parts once more, we obtain

$$-\frac{1}{t^2} \sum_k \int \frac{e^{iktU}}{ik} \partial_y \left(\frac{1}{U'} \partial_y \left(\frac{f_k \psi_{-k}}{U'} \right) \right),$$

and an additional boundary term in the setting of a finite channel:

$$\frac{1}{t^2} \sum \frac{e^{iktU(y)}}{ikU'} \partial_y \left(\frac{f_k \psi_{-k}}{U'} \right) \Big|_{y=0}^{1}.$$

Using Hölder's inequality, trace estimates and that $\frac{1}{U'} \in W^{2,\infty}(\Omega)$, we hence obtain:

$$\|v_2(t)\|^2_{L^2(\Omega)} \lesssim \mathcal{O}(t^{-2}) \|W(t)\|_{H_x^{-1} H_y^2(\Omega)} \|\psi\|_{H^2(\Omega)}. \tag{30}$$

By classic elliptic regularity theory for the Laplacian, $\|\psi\|_{H^2(\Omega)} \lesssim \|v_2\|_{L^2(\Omega)}$. Thus, dividing by $\|v\|_{L^2(\Omega)}$ yields the result. □

REMARK 2.
- *Assuming that $\|W(t)\|_{H_x^{-1} H_y^2}$ is bounded uniformly in t, we hence obtain damping with the optimal algebraic rates. Furthermore, slightly slower decay still holds, if the growth of the norms of $W(t)$ can be adequately controlled. Consider for example the last inequality* (30)*:*

$$\|v_2(t)\|_{L^2} \lesssim \mathcal{O}(t^{-2}) \|W(t)\|_{H_x^{-1} H_y^2}.$$

If $\|W(t)\|_{H_x^{-1} H_y^2}$ grows with a rate of $\mathcal{O}(t^\alpha), \alpha < 2$, then $\|v_2(t)\|_{L^2} = \mathcal{O}(t^{\alpha-2})$ still decays.
- *Analogously to Lemma 2.5, it is possible to interpolate between the two estimates of Theorem 4.4 and hence obtain*

$$\|v(t) - \langle v \rangle_x\|_{L^2(\Omega)} = \mathcal{O}(t^{-s}) \|W(t)\|_{H_x^{-1} H_y^s(\Omega)},$$

for $1 < s < 2$, provided $W(t) \in H_x^{-1} H_y^s(\Omega)$ for all times.

Consider the linearized Euler equations, (19), in either the finite or infinite channel and introduce

$$V_2(t,x,y) := v_2(t, x - tU(y), y).$$

Then W satisfies

$$\partial_t W = U''(y) V_2. \tag{31}$$

Furthermore, since

$$(x,y) \mapsto (x - tU(y), y)$$

is an L^2 isometry,

$$\|V_2\|_{L^2(\Omega)} = \|v_2\|_{L^2(\Omega)}.$$

Integrating (31), sufficient decay of $\|v_2\|_{L^2}$ hence implies a scattering result.

THEOREM 4.5 (Scattering). *Let Ω be either the infinite periodic channel or finite periodic channel and let ω be a solution of the linearized Euler equations,* (19)*, on Ω with initial datum $\omega_0 \in L^2_x H^2_y(\Omega)$. Let further U satisfy the assumptions of Theorem 4.4, $U'' \in L^\infty(\Omega)$ and suppose that, for all times t, W satisfies*

$$\|W - \langle W \rangle_x\|_{H_x^{-1} H_y^2(\Omega)} < C < \infty.$$

Then there exist asymptotic profiles $W^{\pm\infty} \in L^2_x H^2_y(\Omega)$, such that

$$W \xrightarrow{L^2} W^{\pm\infty},$$

as $t \to \pm\infty$.

PROOF. By Duhamel's formula, which in our scattering formulation is just integrating (31), W satisfies

$$W(t) = \omega_0 + \int_0^t U'' V_2(\tau) d\tau. \tag{32}$$

By Theorem 4.4, we control

$$\left\| \int_0^t U'' V_2(\tau) d\tau \right\|_{L^2(\Omega)} \leq \|U''\|_{L^\infty(\Omega)} \int_0^t \mathcal{O}(\tau^{-2}) d\tau.$$

Therefore, the limits $W^{\pm\infty}$ of (32) as $t \to \pm\infty$ exist in $L^2(\Omega)$ and by weak compactness of the unit ball in $H_x^{-1} H_y^2(\Omega)$ and lower semi-continuity, also $W^{\pm\infty} \in H_x^{-1} H_y^2(\Omega)$. □

In the following subsection, we further generalize the conditional damping results from shear flows, $(x, y) \mapsto (x - tU(y), y)$, to diffeomorphisms Y, which are structurally similar to shear flows.

1.1. Diffeomorphisms with shearing structure. Consider the full 2D Euler equations in either the infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, or the finite periodic channel, $\mathbb{T} \times [0, 1]$,

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= 0, \\ \nabla \times v &= \omega, \\ \nabla \cdot v &= 0, \\ \omega|_{t=0} &= \omega_0, \end{aligned} \tag{33}$$

where, in the case of a finite periodic channel, we consider impermeable walls, i.e.

$$v_2 = 0, \text{ for } y \in \{0, 1\}. \tag{34}$$

Restricting to sufficiently regular solutions, by the results of Section 1.4, we may equivalently consider the evolution of the flow maps X_t:

$$\begin{aligned} \partial_t X_t &= v(t, X_t), \\ X_0 &= Id, \\ \omega(t, X_t) &= \omega_0. \end{aligned} \tag{35}$$

We further recall that, as v is divergence-free, DX satisfies

$$\det(DX) \equiv 1,$$

and is thus measure-preserving and invertible. Hence, if $\omega_0 \in L^p(\Omega)$, then for any time t also $\omega(t) \in L^p(\Omega)$ and

$$\|\omega(t)\|_{L^p(\Omega)} = \|\omega_0\|_{L^p(\Omega)}.$$

However, we note that, in an infinite periodic channel, for solutions close to a monotone shear flow, $U(y)$, in general $\omega_0 \notin L^p(\mathbb{T}_L \times \mathbb{R})$, since $\nabla \times (U(y), 0) = -U'(y) \notin L^p(\mathbb{T}_L \times \mathbb{R})$. Furthermore, if X_t is not a shear, then

$$\langle \omega \rangle_x = \langle \omega_0 \circ X \rangle_x \neq \langle \omega_0 \rangle_x \circ X \neq \langle \omega_0 \rangle_x.$$

Thus, unlike in the linear setting, the "underlying shear":

$$\langle v \rangle_x = \begin{pmatrix} \langle v_1 \rangle_x(t,y) \\ 0 \end{pmatrix} \tag{36}$$

corresponding to

$$\begin{aligned} \nabla \times \langle v \rangle_x &= \langle \omega \rangle_x, \\ \nabla \cdot \langle v \rangle_x &= 0, \end{aligned}$$

is not anymore time-independent.

In the following, we thus instead consider $\langle \omega \rangle_x(t,y)$ and $\langle v \rangle_x(t,y)$ as given functions and let Y_t denote the flow by $\langle v \rangle_x$, i.e. the solution map of

$$\partial_t f + \langle v \rangle_x \cdot \nabla f = 0.$$

The flow, Y_t, is then of the form

$$Y_t : (x,y) \mapsto (x - u(t,y), y), \tag{37}$$

where

$$u(t,y) = \int_0^t \langle v_1 \rangle_x(\tau, y) d\tau. \tag{38}$$

In particular, denoting

$$W(t) := (\omega - \langle \omega \rangle_x) \circ Y_t^{-1},$$

we observe that, unlike (36):

$$\langle W(t) \rangle_x = \langle W(t) \rangle_x \circ Y_t = 0. \tag{39}$$

Similar to Theorem 4.4, in the following theorem we *assume* that Y_t is a good approximation to X in the sense that $W(t) \in H^2_{x,y}(\Omega)$, uniformly in time.

We then study under which assumptions on Y_t, the perturbation to the velocity field $v - \langle v \rangle_x$:

$$\begin{aligned} \nabla \times (v - \langle v \rangle_x) &= \omega - \langle \omega \rangle_x = W \circ Y_t, \\ \nabla \cdot (v - \langle v \rangle_x) &= 0, \end{aligned} \tag{40}$$

decays with algebraic rates.

THEOREM 4.6 (Damping in terms of the flow map Y and W). *Let $W(t) \in L^2_x H^1_y(\Omega)$ be such that for all times*

$$\langle W(t) \rangle_x = 0, \tag{41}$$

$$\|W(t)\|_{L^2_x H^1_y(\Omega)} < C < \infty. \tag{42}$$

Let further Y_t be given by

$$Y_t : (x,y) \mapsto (x - u(t,y), y), \tag{43}$$

and suppose $\partial_y u(t,y) \in W^{2,\infty}$ satisfies

$$\inf_{t,y} \frac{1}{t} \partial_y u(t,y) > c > 0. \tag{44}$$

Then, for any test function $\psi \in H^1(\Omega)$ with compact support in y:

(45)

$$\begin{aligned}\iint \psi W \circ Y &= \iint \psi \frac{d}{dx}\left(\frac{d}{dx}\right)^{-1} W \circ Y \\ &= \iint \psi \frac{1}{\partial_y u(t,y)} \frac{d}{dy}\left(\frac{d}{dx}\right)^{-1} W \circ Y + \left(\frac{d}{dx}\right)^{-1} (\partial_y W) \circ Y) \\ &= \iint \frac{1}{\partial_y u(t,y)} \psi \left(\frac{d}{dx}\right)^{-1} ((\partial_y W) \circ Y) - \frac{d}{dy}\left(\frac{1}{\partial_y u(t,y)}\psi\right)\left(\frac{d}{dx}\right)^{-1} W \circ Y.\end{aligned}$$

In particular, taking the supremum over all test functions ψ such that $\|\psi\|_{H^1(\Omega)} \leq 1$, we obtain

$$\|v - \langle v\rangle_x\|_{L^2} \lesssim \frac{1}{ct}\|W\|_{H_x^{-1}H_y^1} \lesssim \frac{1}{ct}\|W\|_{L_x^2 H_y^1} = \mathcal{O}(t^{-1}).$$

Proof of Theorem 4.6. As W satisfies $\langle W\rangle_x = 0$ and as this property is preserved under composition with Y, $\left(\frac{d}{dx}\right)^{-1} W \circ Y$ is well-defined and

$$W \circ Y = \frac{d}{dx}\left(\frac{d}{dx}\right)^{-1} W \circ Y = \frac{d}{dx}\left(\left(\frac{d}{dx}\right)^{-1} W\right) \circ Y.$$

We further note that, by the chain rule

$$\begin{aligned}\frac{d}{dx} W \circ Y &= \partial_x Y_1 (\partial_x W) \circ Y + \partial_x Y_2 (\partial_y W) \circ Y, \\ \frac{d}{dy} W \circ Y &= \partial_y Y_1 (\partial_x W) \circ Y + \partial_y Y_2 (\partial_y W) \circ Y,\end{aligned} \tag{46}$$

and that

$$\det \begin{pmatrix} \partial_x Y_1 & \partial_y Y_1 \\ \partial_x Y_2 & \partial_y Y_2 \end{pmatrix} \equiv 1. \tag{47}$$

Thus,

$$\frac{d}{dx} W \circ Y = \frac{\partial_x Y_2}{\partial_y Y_1} \frac{d}{dy} W \circ Y + \frac{1}{\partial_y Y_1} (\partial_y W) \circ Y.$$

The equation (45) hence follows using integration by parts.

In order to prove the desired damping result, we recall from the proof of Theorem 4.4, that

$$\|v - \langle v\rangle_x\|_{L^2} \lesssim \sup_{\psi : \|\psi\|_{H^1(\Omega)} \leq 1} \iint \psi(\omega - \langle \omega\rangle_x).$$

Using (45), the proof hence concludes by an application of Hölder's inequality and using that

$$\frac{1}{\partial_y u(t,y)} < \frac{1}{ct}.$$

□

As seen in the proof, the theorem can be formulated for flows not of the form (37) and we can also allow $\det(DY)$ to be non-constant. In this case, (45) is replaced by

$$\iint \psi W \circ Y = \iint \frac{\det(DY)}{\partial_y Y_1} \psi \left(\frac{d}{dx}\right)^{-1} ((\partial_y W) \circ Y) - \frac{d}{dy}\left(\frac{\partial_x Y_2}{\partial_y Y_1}\psi\right)\left(\frac{d}{dx}\right)^{-1} W \circ Y.$$

However, in order to use $\left(\frac{d}{dx}\right)^{-1} W \circ Y$, we have to require that

$$\langle W \circ Y \rangle_x = 0,$$

which heavily restricts the possible choices for Y and W. In particular, in general one can not choose $W = \omega_0 - \langle \omega_0 \rangle_x$ and $Y = X$. However, as we will see in Section 4 of Chapter 6, flows of the form (37) are well-suited to study the behavior close to shear flow solutions.

Thus far all damping results have been *conditional* under the assumption of regularity. In the following two sections we remove this restriction by establishing stability and thus regularity of the linearized Euler equations considered as a scattering problem around the underlying transport equation,

$$\partial_t \omega + U(y)\partial_x \omega = 0.$$

2. Asymptotic stability for an infinite channel

As discussed in Section 1, thus far all our damping results are *conditional* under the assumption that our scattered solution, W, of

$$\begin{aligned} \partial_t \omega + U(y)\partial_x \omega &= U'' v_2, \text{ on } \mathbb{T}_L \times \mathbb{R} \times \mathbb{R} \ni (x,y,t), \\ v_2 &= \partial_x \Delta^{-1} \omega, \\ W(t,x,y) :&= \omega(t, x - tU(y), y), \end{aligned} \tag{48}$$

stays regular in the sense that the L^2, H^1 and H^2 norm of W remain uniformly bounded or at least grow very slowly.

In the case of L^2 stability, as discussed in Chapter 1, there are classical stability results due to Rayleigh, [**Ray79**], Fjørtoft, [**Dra02**, page 132], and Arnold, [**Arn66a**]. However, these results use fundamentally different mechanisms, namely orthogonality, cancellation or convexity, while we use mixing by shearing. In particular, our flows are in general not covered by any of these classical stability results. Furthermore, we show that the shearing mechanism is more robust in the sense that it can also be used to derive stability results in higher Sobolev norms.

Before stating the main result, we introduce coordinate transformations, notation and perform a Fourier transform in x to simplify the equation.

As $U : \mathbb{R} \mapsto \mathbb{R}$ is strictly monotone, it is also bijective and invertible. We hence introduce a change of variables, $y \mapsto z = U(y)$, as well as functions

$$\begin{aligned} f(z) :&= U''(U^{-1}(z)), \\ g(z) :&= U'(U^{-1}(z)). \end{aligned} \tag{49}$$

Here, it is convenient to assume that U' is not only bounded from below but also from above so that the change of variables is bilipschitz. For simplicity of notation, we often also assume that $g > 0$, i.e. U is strictly monotonically increasing, but all described results remain valid for strictly monotonically decreasing U as well.

In the new coordinates, the linearized Euler equations are given by

$$\begin{aligned} \partial_t \omega + z\partial_x \omega &= f(z)\partial_x \phi, \\ (\partial_x^2 + (g(z)\partial_z)^2)\phi &= \omega. \end{aligned} \tag{50}$$

The underlying transport structure hence turns into Couette flow, which is particularly useful for computing derivatives and applications of a Fourier transform. As a trade off, the equation for the stream function is not anymore given by the Laplacian. However, the equation is still elliptic *if and only if* g is bounded away from zero, i.e. iff U is strictly monotone.

Changing to a *scattering formulation*, i.e. introducing

$$
\begin{aligned}
W(t,x,z) &:= \omega(t, x - tz, z),\\
\Phi(t,x,z) &:= \phi(t, x - tz, z),
\end{aligned}
\tag{51}
$$

the left-hand-side of (50) simplifies and we obtain

$$
\begin{aligned}
\partial_t W &= f(z)\partial_x \Phi,\\
(\partial_x^2 + (g(z)(\partial_z - t\partial_x))^2)\Phi &= W.
\end{aligned}
$$

We further note that, like Couette flow, the x average $\langle W\rangle_x = \langle \omega\rangle_x$ satisfies

$$
\partial_t \langle W\rangle_x = f(z)\langle \partial_x \Phi\rangle \equiv 0
$$

and is thus conserved. We may therefore subtract $\langle \omega_0\rangle_x$ from ω_0 and assume that

$$
\langle W\rangle_x(t,y) \equiv 0.
$$

As f and g do not depend on x, after a Fourier transform in x the system *decouples* and the frequency k plays the role of a parameter

$$
\begin{aligned}
\partial_t \hat{W} &= f(z) ik \hat{\Phi}, \text{ on } (\mathbb{Z}\setminus\{0\}) \times \mathbb{R} \times \mathbb{R} \ni (k,y,t),\\
(-k^2 + (g(z)(\partial_z - ikt))^2)\hat{\Phi} &= \hat{W}.
\end{aligned}
$$

Furthermore, we adjust the definition of Φ by dividing by k^2, which is well-defined, as we assumed that

$$
\langle W\rangle_x(t,z) = \hat{W}(k=0,t,\eta) \equiv 0.
$$

Relabeling z as y, we thus obtain the following simplified *linearized Euler equations in scattering formulation*:

$$
\begin{aligned}
\partial_t \hat{W} &= \frac{if}{k}\hat{\Phi}, \text{ on } L(\mathbb{Z}\setminus\{0\}) \times \mathbb{R} \times \mathbb{R} \ni (k,y,t),\\
(-1 + (g(\frac{\partial_y}{k} - it))^2)\hat{\Phi} &= \hat{W}.
\end{aligned}
\tag{52}
$$

Our main result of this section is given by the following stability theorem, which is proved in Subsection 2.3.

THEOREM 4.7 (Sobolev stability for the infinite periodic channel). *Let $s \in \mathbb{N}_0$ and $f, g \in W^{s+1,\infty}(\mathbb{R})$ and suppose that there exists $c > 0$, such that*

$$
0 < c < g < c^{-1} < \infty.
$$

Suppose further that

$$
L\|f\|_{W^{s+1,\infty}}
$$

is sufficiently small. Then for all $m \in \mathbb{N}_0$ and $\omega_0 \in H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})$, the solution W of the linearized Euler equations in scattering formulation, (52)*, with initial datum ω_0 satisfies*

$$
\|W(t,x,y)\|_{H^m_x H^s_y} \lesssim \|\omega_0\|_{H^m_x H^s_y}.
$$

REMARK 3. *As* (52) *decouples with respect to k, in our stability results we actually prove that, for any given k,*

$$
\|\hat{W}(t,k,\cdot)\|_{H^s_y} \lesssim \|\hat{\omega}_0\|_{H^s_y}.
$$

The results for $H^m_x H^s_y$ are then obtained by summing in k. In particular, any result for $L^2_x H^s_y$ can be easily shown to also hold for $H^m_x H^s_y$.

In the following sections, we hence consider k as a fixed parameter in (52) *and study the stability of $\hat{W}(t,k,\cdot) \in H^s(\mathbb{R})$.*

Remark 4. *A main difficulty in establishing stability results such as Theorem 4.7 is that the operator*

$$W \mapsto \Phi,$$

interpreted as an operator from L^2 to L^2 does not improve in time, as multiplication by e^{ikty} is a unitary operation. More precisely, for any given k, the operator norm of the solution operator to

$$e^{-ikty}(-k^2 + (g\partial_y)^2)e^{ikty} \tag{53}$$

is independent of time. As a consequence, the uniform damping results of Section 1 necessarily sacrifice regularity in order to obtain uniform decay. In the proof of Theorem 4.7, we therefore have to use the more subtle mode-wise decay, where for each fixed frequency, (k, η), the solution operator of (53) *decays with rate $\mathcal{O}(|\eta - kt|^{-2})$.*

In the following, we first introduce the mechanism of our proof in a simplified setting of a constant coefficient model, for which we can also compute the solution explicitly. Using a perturbation argument, we establish L^2 stability for the general setting in Section 2.2 and subsequently extend the result to higher Sobolev norms in Section 2.3.

2.1. A constant coefficient model. In order to obtain a better understanding of the dynamics of the linearized Euler equations, in the following we consider a simplified model. Here, we formally replace $f(y)$ and $g(y)$ in (52) by constants to recover the decoupling:

$$\begin{aligned}
&\partial_t \Lambda = c\Psi, \text{ on } L(\mathbb{Z} \setminus \{0\}) \times \mathbb{R} \times \mathbb{R} \ni (k, y, t),\\
&(-1 + (\frac{\partial_y}{k} - it)^2)\Psi = \Lambda,\\
&\Lambda|_{t=0} = \hat{\omega}_0(k, \cdot).
\end{aligned} \tag{CC}$$

Here, $c \in \mathbb{C}$ should be thought of as small and not necessarily imaginary. For simplicity of notation, we choose the constant in front of $(\frac{\partial_y}{k} - it)^2$ to be 1. In general, $\min(g^2) > 0$ is the natural choice.

Like the linearized Euler equations in scattering formulation, (52), the model problem, (CC), decouples with respect to k (c.f. Remark 3). In the following, we hence write $\Lambda(t) \in H^s = H^s(\mathbb{R})$ to denote that, for given k,

$$\Lambda(t, k, \cdot) \in H^s(\mathbb{R}).$$

Estimates in the Sobolev spaces $H^m_x H^s_y(\mathbb{T}_L \times \mathbb{R})$ can then be obtained by summing in k.

By our choice of constant coefficients in (CC), the model problem further decouples after a Fourier transform in y and is explicitly solvable:

Theorem 4.8. *Let $\omega_0 \in L^2$, then the solution of the constant coefficient problem,* (CC)*, is given by*

$$\Lambda = \mathcal{F}^{-1} \exp\left(c\left(\arctan(\frac{\eta}{k} - t) - \arctan(\frac{\eta}{k})\right)\right)\mathcal{F}\omega_0. \tag{54}$$

In particular, for any $s \in \mathbb{N}$ such that $\omega_0 \in H^s$, also $\Lambda(t) \in H^s$ and

$$\|\Lambda(t)\|_{H^s} \leq e^{|c|\pi}\|\hat{\omega}_0(k, \cdot)\|_{H^s}$$

uniformly in time.

REMARK 5. *An estimate by $\pi|\Re(c)|$ would of course also be possible in this case. However, dropping the imaginary part of c corresponds to using antisymmetry and orthogonality, which is more difficult to employ in the variable coefficient setting. As we seek to obtain a robust strategy, we therefore limit ourselves to using the shearing mechanism only.*

While the constant coefficient case allows for an explicit solution, in the general case a more indirect proof is required, which we introduce in the following.

The underlying method of our proof is to introduce a weight that decreases at the right places at a large enough rate to counter potential growth. This method of proof is reminiscent of integrating factors in ODE theory and is sometimes called *ghost energy*, [**Ali01**]. Recent applications of similar methods can, for example, be found in a more sophisticated form in the work of [**BM13b**].

For simplicity of notation, in the following we assume that $c > 0$, in order to avoid writing absolute values.

THEOREM 4.9. *Let $c > 0$ and let Λ be the solution of the constant coefficient problem,* (CC)*, with initial data ω_0. Let $C > 0$ and define*

$$E(t) := \langle \Lambda, \mathcal{F}_\eta^{-1} \exp\left(C \arctan(\frac{\eta}{k} - t)\right) \mathcal{F}_y \Lambda\rangle =: \langle \Lambda, A(t)\Lambda\rangle. \tag{55}$$

Then for $|c| \ll C$ sufficiently small, $E(t)$ is non-increasing and uniformly comparable to $\|W(t)\|_{L^2}^2$. In particular:

$$e^{-C\pi}E(t) \leq \|\Lambda(t)\|_{L^2}^2 \leq e^{C\pi}E(t) \leq e^{C\pi}E(0) \leq e^{2C\pi}\|\omega_0\|_{L^2}^2. \tag{56}$$

REMARK 6. *As can be seen from the explicit solution, the assumptions of Theorem 4.9 and the factors in* (56) *are not optimal for our decoupling model. For example, even for large c, choosing $C \geq c$ would work. However, in the general case, we additionally have to control the commutator of A and multiplication by $\frac{f}{ik}$. Hence, at least for finite times, we can not avoid incurring an operator norm, $e^{C\pi}$, and thus a condition of the form*

$$c < Ce^{-C\pi},$$

which does not improve for large C. This is discussed in more detail in Section 2.2. Therefore, we think of C as approximately 1 and require c to be small.

PROOF OF THEOREM 4.9. We compute the time-derivative of $E(t)$:

$$\partial_t E(t) = \langle \Lambda, \dot{A}\Lambda\rangle + 2\Re\langle A(t)\Lambda, c\Psi\rangle. \tag{57}$$

By our choice of A, $\dot{A}$ is a negative semidefinite symmetric operator. For the proof of our theorem it hence suffices to show that

$$\langle \Lambda, \dot{A}\Lambda\rangle \leq 0$$

is negative enough to absorb the possible growth of

$$|2\Re\langle A(t)\Lambda, c\Psi\rangle|.$$

This therefore ensures that $\partial_t E(t) \leq 0$.

Using Plancherel, it suffices to show that

$$\int_{\mathbb{R}} \frac{-Ce^{C\arctan(\frac{\eta}{k}-t)}}{1 + (\frac{\eta}{k} - t)^2}|\tilde{\Lambda}(t, k, \eta)|^2 d\eta + 2\int_{\mathbb{R}} \Re(c)\frac{e^{C\arctan(\frac{\eta}{k}-t)}}{1 + (\frac{\eta}{k} - t)^2}|\tilde{\Lambda}(t, k, \eta)|^2 d\eta \leq 0, \tag{58}$$

for arbitrary functions $|\tilde{\Lambda}(t, k, \eta)|$, which in this case holds if

$$2|c| \leq C.$$

□

2.2. L^2 stability for monotone shear flows. In the following, we adapt the L^2 stability result, Theorem 4.9 of Section 2.1, to the linearized Euler equations in scattering formulation, (52),

$$
\begin{aligned}
&\partial_t W = \frac{if}{k}\Phi, \\
(59) \qquad &(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi = W, \\
&(k, y, t) \in L(\mathbb{Z} \setminus \{0\}) \times \mathbb{R} \times \mathbb{R},
\end{aligned}
$$

where for simplicity we dropped the hats, $\hat{\cdot}$, from our notation. As noted in Remark 3, (59) decouples with respect to k. For the remainder of this chapter and also in the following chapter, we thus follow the same convention as in Section 2.1 and use $W \in H^s(\mathbb{R})$ to denote that, for given k,

$$W(t, k, \cdot) \in H^s(\mathbb{R}).$$

In analogy to the constant coefficient model, (CC), for a given solution W of (59), we introduce the *constant coefficient stream function* Ψ:

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi = W. \tag{60}$$

We stress that, starting from this section, Ψ does not correspond to a solution of the constant coefficient problem, (CC), but only to a given right-hand-side W in (60).

More generally, we introduce the following notation:

Definition 4.1 (Constant coefficient stream function). Let $k \in L(\mathbb{Z} \setminus \{0\})$ and let $R(t) \in L^2(\mathbb{R})$ be a given function. Then the *constant coefficient stream function*, $\Psi[R](t)$, is defined as the solution of

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[R](t, y) = R(t, y). \tag{61}$$

Let further W be a solution of (59), then for any k, t

$$\Psi(t, k, y) := \Psi[W(t, k, \cdot)](t, y). \tag{62}$$

As Φ and $\Psi = \Psi[W]$ satisfy very similar (shifted elliptic) equations, (59) and (60), with the same right-hand-side, we can estimate Sobolev norms of Φ in terms of Ψ, as is shown in Lemma 4.1. We note that, for this purpose, W need not solve (59), but can be any given L^2 function.

Lemma 4.1. *Let $\frac{1}{g} \in W^{1,\infty}$ and assume there exists $c > 0$ such that*

$$0 < c < g < c^{-1} < \infty.$$

Then for any $W(t) \in L^2(\mathbb{R})$, the solutions Φ, Ψ of

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi = W, \tag{63}$$

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi = W, \tag{64}$$

satisfy

$$\|\Phi\|_{\tilde{H}^1}^2 := \|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2 \lesssim \|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2.$$

In the following, we establish L^2 stability of (59) using Lemma 4.1 and subsequently give a proof of Lemma 4.1.

THEOREM 4.10 (L^2 stability for the infinite periodic channel). *Let W be a solution to the linearized Euler equations,* (59)*, and assume that g satisfies the assumptions of Lemma 4.1. Let further A be defined as in Theorem 4.9, i.e.*

$$I(t) := \langle W, A(t)W\rangle_{L^2(\mathbb{R})} := \int |\tilde{W}(t,k,\eta)|^2 \exp\left(C \arctan(\frac{\eta}{k} - t)\right) d\eta, \tag{65}$$

and suppose that

$$\|f\|_{W^{1,\infty}} L$$

is sufficiently small. Then, for any initial datum $\omega_0 \in L^2(\mathbb{R})$, $I(t)$ is non-increasing and satisfies

$$\|W(t)\|_{L^2}^2 \lesssim I(t) \leq I(0) \lesssim \|\omega_0\|_{L^2}^2.$$

PROOF OF THEOREM 4.10. Let $\Psi[AW]$ be as in Definition 4.1, i.e. $\Psi[Aw] \in L^2$ is the solution of

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[AW] = AW.$$

Then, by integration by parts, the time-derivative of $I(t)$ satisfies

$$\begin{aligned} \partial_t I(t) &= \langle W, \dot{A}W\rangle + 2\Re\langle AW, \frac{if}{k}\Phi\rangle \\ &\leq \langle W, \dot{A}W\rangle + 2\|\frac{f}{k}\|_{W^{1,\infty}} \|\Psi[AW]\|_{\tilde{H}^1}\|\Phi\|_{\tilde{H}^1}. \end{aligned} \tag{66}$$

By Lemma 4.1, the last term is further controlled by

$$C_1 \|\frac{f}{k}\|_{W^{1,\infty}} \|\Psi[AW]\|_{\tilde{H}^1}\|\Psi\|_{\tilde{H}^1}. \tag{67}$$

As A is a bounded Fourier multiplier and commutes with the Fourier multiplier $u \mapsto \Psi[u]$, we control

$$\|\Psi[AW]\|_{\tilde{H}^1} \leq \|A\|\|\Psi\|_{\tilde{H}^1} \leq \|A\|\sqrt{|\langle W, \Psi\rangle|}, \tag{68}$$

where we used that

$$\|\Psi\|_{\tilde{H}^1}^2 := \|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2 = -\langle W, \Psi\rangle. \tag{69}$$

Furthermore,

$$\begin{aligned} -\langle W, A\Psi\rangle &= -\langle(-k^2 + (\frac{\partial_y}{ik} - t)^2)\Psi, A\Psi\rangle \\ &= \int (k^2 + (\frac{\eta}{k} - t)^2) \exp\left(C\arctan(\frac{\eta}{k} - t)\right) |\tilde{\Psi}(t,k,\eta)|^2 d\eta. \end{aligned} \tag{70}$$

Therefore,

$$\|\Psi\|_{\tilde{H}^1}^2 \leq \|A\|(-\langle W, A\Psi\rangle) \leq \|A\|^2\|\Psi\|_{\tilde{H}^1}^2,$$

where we used that A^{-1} has the same operator norm as A. Thus, (67) is further controlled by

$$C_1\|\frac{f}{k}\|_{W^{1,\infty}}\|A\|^2|\langle W, A\Psi\rangle|. \tag{71}$$

Hence, combining (66) and (71), $I(t)$ satisfies

$$\partial_t I(t) \leq \langle W, \dot{A}W\rangle + C_2\|A\|^2\|\frac{f}{k}\|_{W^{1,\infty}}|\langle W, A\Psi[W]\rangle|.$$

Using the explicit characterization of A and Ψ in Fourier space, we conclude as in the proof of Theorem 4.9, provided

$$c := C_2\|f\|_{W^{1,\infty}}\|A\|^2 \sup_{k\neq 0}\frac{1}{|k|} \lesssim e^{2C\pi}\|f\|_{W^{1,\infty}} L \tag{72}$$

is sufficiently small. $\square$

Proof of Lemma 4.1. Testing (63) with $\frac{1}{g}\Phi$ and integrating by parts, we obtain:

$$\int \frac{1}{g}|\Phi|^2 + g|(\frac{\partial_y}{k} - it)\Phi|^2 = \langle W, \frac{1}{g}\Phi\rangle. \tag{73}$$

As by our assumption, $c < g < c^{-1}$, the left-hand-side is bounded from below by

$$c(\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2) \gtrsim \|\Phi\|_{\tilde{H}^1}^2.$$

Hence, it remains to estimate $\langle W, \frac{1}{g}\Phi\rangle$ from above.

Using (64) and integrating by parts, we obtain

$$\begin{aligned}\langle W, \frac{1}{g}\Phi\rangle &= \left\langle (-1 + (\frac{\partial_y}{k} - it)^2)\Psi, \frac{1}{g}\Phi \right\rangle \\ &\leq \sqrt{\|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2}\sqrt{\|\frac{1}{g}\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\frac{1}{g}\Phi\|_{L^2}^2} \\ &\lesssim \|\frac{1}{g}\|_{W^{1,\infty}}\sqrt{\|\Psi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}^2}\sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2}.\end{aligned}$$

Dividing by $\|\Phi\|_{\tilde{H}^1} = \sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2}$, we thus obtain the result. $\square$

Remark 7. *Testing* (63) *with* Φ *instead of* $\frac{1}{g}\Phi$ *has the small drawback of introducing commutators involving* gg' *on the left-hand-side, which one can control either by a smallness or sign condition. The right-hand-side however is simplified.*

Testing (64) *with* Ψ *and integrating*

$$\langle W, \Psi\rangle = \langle (-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi, \Psi\rangle \tag{74}$$

by parts, we analogously obtain that

$$\|\Psi\|_{\tilde{H}^1} \lesssim \|\Phi\|_{\tilde{H}^1}.$$

One can more generally show that, up to a factor, both Φ *and* Ψ *attain*

$$\|W\|_{\tilde{H}^{-1}} := \sup\{\langle W, \mu\rangle : \|\mu\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\mu\|_{L^2}^2 \leq 1\}.$$

Remark 8. *It is possible to reduce the requirements of Theorem 4.10 for large* $\|A\|$ *slightly, by noting that*

$$\Psi[AW] = A\Psi[W],$$

as Fourier multipliers commute and that, as a positive multiplier, we can split $A = A^{1/2}A^{1/2}$ *for the purpose of our* L^2 *bound. Hence, in* (66)*, instead of estimating*

$$2\Re\langle AW, \frac{if}{k}\Phi\rangle \lesssim C_1\|\frac{f}{k}\|_{W^{1,\infty}}\|A\|^2|\langle W, A\Psi\rangle|,$$

it suffices to obtain an estimate of the form

$$\|A^{1/2}\frac{if}{k}\Phi\|_{\tilde{H}^1} \lesssim \|A^{1/2}\Psi\|_{\tilde{H}^1}.$$

However, we note that for non-constant f*, even for* $\Phi = \Psi$*, such an estimate would have to control*

$$A^{1/2}fA^{-1/2}, \tag{75}$$

as an operator from $\tilde{H}^1$ *to* $\tilde{H}^1$. *Asymptotically, i.e. for* $t \to \pm\infty$, $\arctan(\eta - t) \to \pm\frac{\pi}{2}$ *and thus* $A^{\pm 1} \rightharpoonup e^{\pm C \frac{\pi}{2}} Id$. *Therefore, for all* C,

$$A^{1/2} f A^{-1/2} \rightharpoonup f,$$

as $t \to \pm\infty$. *However, for each finite time we obtain commutators involving*

$$C(\arctan(\eta_1 - t) - \arctan(\eta_2 - t)),$$

which are not bounded uniformly in C. *Hence, at least for finite times, the operator norm corresponding to* (75) *is not better than*

$$e^{c_1 C} \|f\|_{W^{1,\infty}}.$$

for some $c_1 > 0$, *and thus only provides a small improvement over* (72).

2.3. Iteration to arbitrary Sobolev norms. Thus far we have only shown L^2 stability. In order to derive damping, it remains to extend the result to ensure stability in higher Sobolev norms.

In the constant coefficient model, this generalization is trivial as our equation is invariant under taking derivatives. Hence, after relabeling, we may apply the L^2 result to $\partial_y^s \Lambda$.

Corollary 4.1. *Let* $s \in \mathbb{N}$, $\omega_0 \in H^s(\mathbb{R})$ *and let* Λ *be the solution of the constant coefficient problem,* (CC), *with initial data* ω_0. *Then* $\partial_y^s \Lambda$ *solves the constant coefficient problem,* (CC), *with initial data* $\partial_y^s \omega_0$ *and for* $c < Ce^{-C\pi}$,

$$\|\partial_y^s \Lambda\|_{L^2} \lesssim \|\partial_y^s \omega_0\|_{L^2}.$$

When taking derivatives of the linearized Euler equations, we obtain additional lower order corrections due to commutators. More precisely, for given $j \in \mathbb{N}$, $\partial_y^j W$ satisfies:

$$\begin{aligned} \partial_t \partial_y^j W &= \frac{i}{k} \partial_y^j (f\Phi) =: \frac{i}{k} \sum_{j' \leq j} c_{jj'} (\partial_y^{j-j'} f) \partial_y^{j'} \Phi, \\ (-1 + (g(\frac{\partial_y}{k} - it))^2) \partial_y^{j'} \Phi &= \partial_y^{j'} W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^{j'}]\Phi. \end{aligned} \tag{76}$$

In order to control these corrections, we introduce a family of energies

$$I_j(t) = \langle \partial_y^j W, A \partial_y^j W \rangle, \tag{77}$$

and a combined energy:

$$E_j(t) = \sum_{j' \leq j} I_{j'}(t). \tag{78}$$

With this notation our main theorem is:

Theorem 4.11 (Soboloev stability for the infinite periodic channel). *Let* $j \in \mathbb{N}$ *and assume* f, g *satisfy the assumptions of Theorem 4.10,* $f, g \in W^{j+1,\infty}(\mathbb{R})$ *and that*

$$\|f\|_{W^{j+1,\infty}} L$$

is sufficiently small. Then for any initial datum $\omega_0 \in H^j(\mathbb{R})$, $E_j(t)$ *is non-increasing and satisfies*

$$\|W(t)\|_{H^j}^2 \lesssim E_j(t) \leq E_j(0) \lesssim \|\omega_0\|_{H^j}^2.$$

As in the previous proof, we compare with constant coefficient potentials Ψ:

Lemma 4.2. *Let* $j \in \mathbb{N}$ *and let* g *satisfy the assumptions of Theorem 4.11. Then,*

$$\|\partial_y^j \Phi\|_{\tilde{H}^1} \lesssim \sum_{j' \leq j} \|\partial_y^{j'} \Psi\|_{\tilde{H}^1}.$$

PROOF OF THEOREM 4.11. For any $j' \leq j$, $I_{j'}$ satisfies

$$(79)\quad \begin{aligned} \partial_t I_{j'}(t) &= \langle \partial_y^{j'} W, \dot{A}\partial_y^{j'} W\rangle + \langle A\partial_y^{j'} W, \partial_y^{j'} \frac{if}{k}\Phi\rangle \\ &\leq \langle \partial_y^{j'} W, \dot{A}\partial_y^{j'} W\rangle + \|\Psi[A\partial_y^{j'} W]\|_{\tilde{H}^1} \left\|\frac{f}{k}\right\|_{W^{j'+1,\infty}} \sum_{j''\leq j'} \|\partial_y^{j''}\Phi\|_{\tilde{H}^1}. \end{aligned}$$

Summing over all $j' \leq j$ and using Lemma 4.2 and Young's inequality, we hence obtain:

(80)

$$\begin{aligned} \partial_t E_j(t) &\leq \sum_{j'\leq j} \langle \partial_y^{j'} W, \dot{A}\partial_y^{j'} W\rangle + \left\|\frac{f}{k}\right\|_{W^{j+1,\infty}} \left(\sum_{j'\leq j} \|\Psi[A\partial_y^{j'} W]\|_{\tilde{H}^1}^2 + \|\partial_y^{j'}\Phi\|_{\tilde{H}^1}^2\right) \\ &\lesssim \sum_{j'\leq j} \langle \partial_y^{j'} W, \dot{A}\partial_y^{j'} W\rangle + \left\|\frac{f}{k}\right\|_{W^{j+1,\infty}} \left(\sum_{j'\leq j} \|\Psi[A\partial_y^{j'} W]\|_{\tilde{H}^1}^2 + \|\partial_y^{j'}\Psi\|_{\tilde{H}^1}^2\right). \end{aligned}$$

We further note that $\partial_y^{j'}\Psi = \Psi[\partial_y^{j'} W]$. Hence, relabeling and applying the constant coefficient L^2 result, Theorem 4.9, we obtain that for any j' and for c sufficiently small

$$(81)\quad \langle \partial_y^{j'} W, \dot{A}\partial_y^{j'} W\rangle + c(\|\Psi[\partial_y^{j'} W]\|_{\tilde{H}^1}^2 + \|\Psi[A\partial_y^{j'} W]\|_{\tilde{H}^1}^2) \leq 0.$$

Supposing that

$$\sup_{k\neq 0} \|\frac{f}{k}\|_{W^{j+1,\infty}} = \|f\|_{W^{j+1,\infty}} L \ll c,$$

summing (81) with respect to j and (80) hence imply

$$(82)\quad \partial_t E_j(t) \leq 0,$$

which concludes our proof. □

PROOF OF LEMMA 4.2. We prove the result by induction in j. The case $j = 0$ has been proven as Lemma 4.1 in Section 2.2. Hence, it suffices to show the induction step $j - 1 \mapsto j$:

$$(83)\quad \|\partial_y^j \Phi\|_{\tilde{H}^1} \lesssim \|\partial_y^j \Psi\|_{\tilde{H}^1} + \sum_{j'\leq j-1} \|\partial_y^{j'}\Phi\|_{\tilde{H}^1},$$

for $j \geq 1$.

Recall that $\partial_y^j\Phi$ satisfies (76):

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)\partial_y^j\Phi = \partial_y^j W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^j]\Phi.$$

Proceeding as in the proof of Lemma 4.1, we thus test (76) with $\frac{1}{g}\partial_y^j\Phi$ to obtain an estimate by

$$(84)\quad \|\partial_y^j\Phi\|_{\tilde{H}^1}^2 \lesssim \|\partial_y^j\Psi\|_{\tilde{H}^1}\|\partial_y^j\Phi\|_{\tilde{H}^1} + \langle \partial_y^j\Phi, [(g(\frac{\partial_y}{k} - it))^2, \partial_y^j]\Phi\rangle,$$

where we used that

$$(85)\quad |\langle \partial_y^j\Phi, \partial_y^j W\rangle| = |\langle \partial_y^j\Phi, (-1 + (\frac{\partial_y}{k} - it)^2)\partial_y^j\Psi\rangle| \leq \|\partial_y^j\Psi\|_{\tilde{H}^1}\|\partial_y^j\Phi\|_{\tilde{H}^1}.$$

In order to estimate the contribution of the commutator,

$$(86)\quad [(g(\frac{\partial_y}{k} - it))^2, \partial_y^j]\Phi,$$

we note that at least one of the derivatives ∂_y^j has to fall on the coefficient function g. Hence, (86) can be expressed in terms of

$$\partial_y^{j'}\Phi,\ (g(\frac{\partial_y}{k}-it))\partial_y^{j'}\Phi$$

and

$$(g(\frac{\partial_y}{k}-it))^2\partial_y^{j'}\Phi, \tag{87}$$

with $j' \leq j-1$. Integrating $(\frac{\partial_y}{k}-it)$ by parts in the case (87), (84) is thus further estimated by

$$\|\partial_y^j\Phi\|_{\tilde{H}^1}^2 \lesssim \|\partial_y^j\Psi\|_{\tilde{H}^1}\|\partial_y^j\Phi\|_{\tilde{H}^1} + C(g)\|\partial_y^j\Phi\|_{\tilde{H}^1}\sum_{j'\leq j-1}\|\partial_y^{j'}\Phi\|_{\tilde{H}^1}, \tag{88}$$

where $C(g)$ depends on all derivatives of g up to order j.

Dividing (88) by $\|\partial_y^j\Phi\|_{\tilde{H}^1}$ hence proves the induction step, (83), and concludes our proof. □

As we discuss in Section 4, Theorem 4.11 in particular provides a uniform control of

$$\|W\|_{L_x^2H_y^2(\mathbb{T}_L\times\mathbb{R})},$$

and hence allows us close our strategy and thus prove linear inviscid damping with the optimal decay rates for a large class of monotone shear flows in an infinite periodic channel. Furthermore, as discussed in Section 1, as a consequence of sufficiently fast damping, we obtain a scattering result via Duhamel's formula.

Prior to this, however, we in the next Section 3 prove a similar stability result in the case of a finite channel $\mathbb{T}_L\times[0,1]$ with impermeable walls. There, boundary effects are shown to have a non-negligible effect on the dynamics.

3. Asymptotic stability for a finite channel

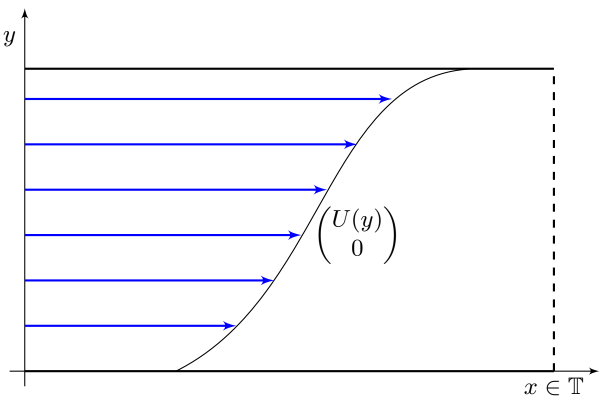
Inspired by the Fourier proof in the whole space case, in the following we establish stability in the setting of a finite periodic channel $\mathbb{T}_L\times[a,b]$. The physically natural boundary conditions in this setting are that the boundary in y is impermeable

$$v_2 = 0, \quad \text{for } y\in\{a,b\}. \tag{89}$$

As the stream function ϕ satisfies

$$v_2 = \partial_x\phi,$$

this, in particular, implies that ϕ restricted to the boundary only depends on time.



Following the same reduction steps as in Section 2.1, in particular removing the mean $\langle W \rangle_x$, ϕ and thus Φ vanishes identically on the boundary. The linearized Euler equations in scattering formulation are hence given by

$$
\begin{aligned}
\partial_t W &= \frac{if(y)}{k}\Phi, \\
(-1 + (g(y)(\frac{\partial_y}{k} - it))^2)\Phi &= W, \\
\Phi|_{y=U(a),U(b)} &= 0, \\
(t,k,y) &\in \mathbb{R} \times L(\mathbb{Z} \setminus \{0\}) \times [U(a), U(b)].
\end{aligned}
\tag{90}
$$

In order to simplify notation, we translate in y and rescale L by a factor (see Section 1.3 of chapter 1) to reduce to $[U(a), U(b)] = [0,1]$.

As in Section 2 (c.f. Remark 3), the equations (90) decouple with respect to k. Hence, in the following we again consider k as a given parameter and write $W(t) \in H^s$ to denote that

$$W(t,k,\cdot) \in H^s([0,1]).$$

Our main result is given by the following theorem and proved in Section 3.3.

THEOREM 4.12. *Let W be a solution of* (90), *$f, g \in W^{3,\infty}$ and suppose that there exists $c > 0$ such that*

$$0 < c < g < c^{-1} < \infty.$$

Suppose further that

$$\|f\|_{W^{1,\infty}} L$$

is sufficiently small.

Then, for any $\omega_0 \in H^2([0,1])$ with $\omega_0|_{y=0,1} = 0$ and for any time t,

$$\|W(t)\|_{H^2} \lesssim \|\omega_0\|_{H^2}.$$

As we show in the following, the case of a finite channel is not only technically more involved, due to the lack of Fourier methods as well as the loss of the multiplier structure for Φ (even for Couette flow), but the qualitative behavior also changes due to boundary effects.

When differentiating the equation, we see that $\partial_y^n \Phi$ satisfies non-zero Dirichlet boundary conditions. Computing the boundary conditions explicitly, we, in particular, show asymptotic H^2 stability is possible if and only if ω_0 satisfy zero Dirichlet conditions, $\omega_0|_{y=0,1} = 0$. Higher Sobolev norms in turn would require even stronger conditions, as we discuss in Section 6. As the damping results provide the sharp algebraic decay rates already for H^2 regularity, we restrict ourselves to considering only L^2, H^1 and H^2 stability. In Chapter 5, we improve these results to the (almost) sharp Sobolev spaces $H^{3/2-\epsilon}$ for general perturbations and $H^{5/2-\epsilon}$ for perturbations ω_0 with vanishing Dirichlet boundary data, $\omega_0|_{y=0,1} = 0$, respectively.

3.1. L^2 stability via shearing. As in Section 2.2, we consider the linearized Euler equations, (90), this time in the finite periodic channel, $\mathbb{T}_L \times [0,1]$,

$$
\begin{aligned}
\partial_t W &= \frac{if(y)}{k}\Phi, \\
\left(-1 + \left(g(y)\left(\frac{\partial_y}{k} - it\right)\right)^2\right)\Phi &= W, \\
\Phi|_{y=0,1} &= 0, \\
(t,k,y) &\in \mathbb{R} \times L(\mathbb{Z} \setminus \{0\}) \times [0,1],
\end{aligned}
\tag{91}
$$

and additionally introduce the constant coefficient stream function Ψ

$$\begin{aligned}\left(-1+\left(\frac{\partial_y}{k}-it\right)^2\right)\Psi &= W, \\ \Psi_{y=0,1} &= 0.\end{aligned} \tag{92}$$

As in Definition 4.1 of Section 2.2, we introduce constant coefficient stream functions for a given right-hand-side, where additionally prescribe boundary conditions:

Definition 4.2 (Constant coefficient stream function for a finite periodic channel). Let $k \in L(\mathbb{Z} \setminus \{0\})$ and let $R(t) \in L^2([0,1])$ be a given function. Then the *constant coefficient stream function*, $\Psi[R](t)$, is defined as the solution of

$$\begin{aligned}(-1+(\frac{\partial_y}{k}-it)^2)\Psi[R](t,y) &= R(t,y), \\ \Psi[R](t,y)|_{y=0,1} &= 0.\end{aligned} \tag{93}$$

Let further W be a solution of (91), then for any k, t, we define

$$\Psi(t,k,y) := \Psi[W(t,k,\cdot)](t,y). \tag{94}$$

If we considered periodic boundary conditions, in a Fourier expansion, $\Psi[\cdot]$ would again be given by a multiplier and could be estimated explicitly in the same way as in the setting of an infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$. As we however have zero Dirichlet conditions, we can not anymore solve the evolution of a constant coefficient model explicitly, but rather have to establish control of boundary effects and growth of norms, using more indirect methods. Thus, stability results are already non-trivial even for a constant coefficient model.

Emulating the proof of the L^2 stability with a decreasing weight $A(t)$ as in Section 2.2, a natural replacement for the Fourier transform is given by the expansion in an L^2-basis (e_n).

In view of our zero Dirichlet conditions a natural choice of such a basis is

$$\sin(ny), n \in \mathbb{N}.$$

For the current purpose of L^2 stability, however, it is advantageous to instead consider an expansion in the Fourier basis

$$e^{iny}, n \in 2\mathbb{Z},$$

for which calculations greatly simplify, at the cost of worse mapping properties in higher Sobolev spaces. This trade-off and the role of the choice of basis is discussed in more detail in Section 5.

In the following we introduce several lemmata, which allow us to prove L^2 stability in Theorem 4.13:

- Lemma 4.5 provides a definition of a decreasing weight A, as in Theorem 4.9, and proves that the constant coefficient stream function Ψ can be controlled in terms of this weight. In the case of an infinite channel as in Section 2, this result immediately followed from the explicit Fourier characterization. In the setting of a finite channel, however, additional boundary effects have to be controlled, which is accomplished by the basis computations in Lemmata 4.3 and 4.4.
- Lemma 4.6 provides an estimate of Φ in terms of Ψ and hence a reduction similar to Lemma 4.1 of Section 2.2.

LEMMA 4.3. *Let $n \in 2\pi\mathbb{Z}$ and let $\Psi[e^{iny}]$ be given by Definition 4.2, i.e. let $\Psi[e^{iny}]$ be the solution of*

$$\begin{aligned}(-k^2+(\partial_y-ikt)^2)\Psi[e^{iny}]&=e^{iny},\\ \Psi[e^{iny}]|_{y=0,1}&=0.\end{aligned}$$

Then, for any $m \in 2\pi\mathbb{Z}$,

$$\langle\Psi[e^{iny}],e^{imy}\rangle=\frac{\delta_{nm}}{k^2+(n-kt)^2}+\frac{k}{(k^2+(m-kt)^2)(k^2+(n-kt)^2)}(a-b),$$

where a, b solve

$$\begin{pmatrix}e^{k+ikt} & e^{-k+ikt}\\ 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=-\begin{pmatrix}1\\ 1\end{pmatrix}.$$

LEMMA 4.4. *Let $\Psi, W \in L^2$ solve*

$$\begin{aligned}(-k^2+(\partial_y-ikt)^2)\Psi&=W,\\ \Psi|_{y=0,1}&=0.\end{aligned}$$

Denote the basis expansion of W with respect to e^{iny}, $n \in 2\pi\mathbb{Z}$, by

$$W(y)=\sum_n W_n e^{iny}.$$

Then W satisfies

$$|\langle W,\Psi\rangle|\lesssim k^{-2}\sum_n <\frac{n}{k}-t>^{-2}|W_n|^2.$$

LEMMA 4.5. *Define the operator $A(t)$ by*

$$A(t):e^{iny}\mapsto\exp\left(-\int^t<\frac{n}{k}-\tau>^{-2}d\tau\right)e^{iny}=\exp\left(\arctan\left(\frac{n}{k}-t\right)\right)e^{iny}.$$

Then $A: L^2 \to L^2$ is a uniformly bounded, symmetric, positive operator and satisfies

$$\|W\|_{L^2}^2\lesssim\langle W,AW\rangle\lesssim\|W\|_{L^2}^2,$$

where the estimates are uniform in t. Furthermore, the time derivative $\dot{A}$ is symmetric and non-positive and there exists a constant $C>0$ such that, for Ψ as in Definition 4.2,

$$|\langle W,A\Psi\rangle|C+\langle W,\dot{A}W\rangle\leq 0.$$

LEMMA 4.6. *Let $W \in L^2$, $0<c<g<c^{-1}<\infty$, $\frac{1}{g(y)}, f(y) \in W^{1,\infty}$ and A as in Lemma 4.5. Let W, Φ, Ψ solve*

$$\begin{aligned}(k^2+(g(\partial_y-ikt))^2)\Phi&=W,\\ \Phi_{y=0,1}&=0,\end{aligned}\tag{95}$$

$$\begin{aligned}(k^2+(\partial_y-ikt)^2)\Psi&=W,\\ \Psi_{y=0,1}&=0.\end{aligned}\tag{96}$$

Then there exists a constant C such that

$$|\langle AW,\frac{if}{k}\Phi\rangle|\leq\frac{C}{k}|\langle AW,\Psi\rangle|.$$

With these lemmata we can now prove L^2 stability:

THEOREM 4.13 (L^2 stability for the infinite periodic channel). *Let $f, g \in W^{1,\infty}$ and suppose that there exists $c > 0$, such that*

$$0 < c < g < c^{-1} < \infty.$$

Suppose further that

$$L\|f\|_{W^{1,\infty}}$$

is sufficiently small. Then for all $\omega_0 \in L^2$, the solution W of the linearized Euler equations, (90)*, with initial datum ω_0, for any time t, satisfies*

$$\|W(t)\|_{L^2} \lesssim \|\omega_0\|_{L^2}.$$

PROOF OF THEOREM 4.13. The time derivative of $I(t) := \langle W, AW\rangle$ is controlled by

$$2|\langle AW, \frac{if}{k}\Phi\rangle| + \langle W, \dot{A}W\rangle.$$

By Lemma 4.6 there exists a constant C_1, such that

$$\dot{I}(t) \leq \frac{C_1}{|k|}|\langle AW, \Psi\rangle| + \langle W, \dot{A}W\rangle.$$

Requiring $|k|$ to be sufficiently large, $\frac{C_1}{|k|} \leq C$. Thus, Lemma 4.5 yields

$$\dot{I}(t) \leq |\langle W, A\Psi[W]\rangle|C + \langle W, \dot{A}W\rangle \leq 0.$$

In particular,

$$\|W(t)\|_{L^2}^2 \lesssim I(t) \leq I(0) \lesssim \|\omega_0\|_{L^2}^2.$$

□

It remains to prove the previously stated Lemmata 4.3-4.6.

PROOF OF LEMMA 4.3. The constant coefficient stream function for e^{iny} is given by

$$\Psi[e^{iny}] = \frac{1}{k^2 + (n-kt)^2}(e^{iny} + ae^{ky+ikty} + be^{-ky+ikty}),$$

where a, b solve

$$\begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Integrating against another basis function e^{imy}, we obtain:

$$\begin{aligned}
\langle \Psi[e^{iny}], e^{imy}\rangle &= \frac{1}{k^2+(n-kt)^2}\Big(\delta_{nm} + \frac{e^{ky+ikty}\big|_{y=0}^1}{k+i(kt-m)}a + \frac{e^{ky+ikty}\big|_{y=0}^1}{-k+i(kt-m)}\Big) \\
&= \frac{1}{k^2+(n-kt)^2}\Big(\delta_{nm} + \frac{k\ e^{ky+ikty}\big|_{y=0}^1}{k^2+(kt-m)^2}a - \frac{k\ e^{ky+ikty}\big|_{y=0}^1}{k^2+(kt-m)^2}b \\
&\quad - \frac{i(kt-m)}{k^2+(kt-m)^2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\Big) \\
&= \frac{\delta_{nm}}{k^2+(n-kt)^2} + \frac{k}{(k^2+(m-kt)^2)(k^2+(n-kt)^2)}(a-b).
\end{aligned}$$

□

Proof of Lemma 4.4. Using Lemma 4.3, we expand $\langle W, \Psi\rangle$ in our basis and explicitly compute:

$$
\begin{aligned}
\langle W, \Psi\rangle &= \sum_{n,m} \overline{W}_m \langle e^{imy}, \Psi[e^{iny}]\rangle W_n \\
&= \sum_{n,m} \overline{W}_m \left(\frac{\delta_{nm}}{k^2+(n-kt)^2} + \frac{k}{(k^2+(m-kt)^2)(k^2+(n-kt)^2)}(a-b) \right) W_n \\
&= \sum_n \frac{1}{k^2+(n-kt)^2}|W_n|^2 + k(a-b)\left(\sum_n \frac{1}{k^2+(n-kt)^2}W_n\right)\left(\sum_m \frac{1}{k^2+(m-kt)^2}\overline{W}_m\right) \\
&\le \left(\sum_n \frac{|W_n|^2}{k^2+(n-kt)^2}\right)\left(1+|k(a-b)|\left\|\frac{1}{\sqrt{k^2+(m-kt)^2}}\right\|_{l^2_m}^2\right) \\
&\lesssim \sum_n \frac{|W_n|^2}{k^2+(n-kt)^2}.
\end{aligned}
$$

□

Proof of Lemma 4.5. Expressed in the Fourier basis, e^{iny}, $A(t)$ is a diagonal operator with positive, monotonically decreasing coefficients that are uniformly bounded from above and below by $\exp(\pm\| < t >^{-2} \|_{L^1_t}) = e^{\pm\pi}$. It remains to show

$$
|\langle W, A\Psi[W]\rangle| C + \langle W, \dot{A}W\rangle \le 0.
$$

Modifying the proof of Lemma 4.4 slightly, we obtain that

$$
\begin{aligned}
|\langle W, A\Psi[W]\rangle| &\lesssim \sum < \frac{n}{k} - t >^{-2} |W_n|^2 \\
&\lesssim \sum < \frac{n}{k} - t >^{-2} \exp\left(\int^t < \frac{n}{k} - \tau >^{-2} d\tau\right)|W_n|^2 \\
&= -\langle W, \dot{A}W\rangle.
\end{aligned}
$$

□

Proof of Lemma 4.6. Let $\Psi[AW]$ solve

$$
\begin{aligned}
(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[AW] &= AW, \\
\Psi[AW]_{y=0,1} &= 0.
\end{aligned}
$$

By integration by parts, we then obtain

$$
\begin{aligned}
|\langle AW, \frac{if}{k}\Phi\rangle| &\le \sqrt{\|\Psi[AW]\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi[AW]\|_{L^2}^2}\|f\|_{W^{1,\infty}}\frac{1}{|k|}\sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2} \\
&=: \|\Psi[AW]\|_{\tilde{H}^1}\|f\|_{W^{1,\infty}}\frac{1}{|k|}\|\Phi\|_{\tilde{H}^1}.
\end{aligned}
$$

By our basis characterization, and as A is a bounded, positive multiplier on our basis,

$$
\|\Psi[AW]\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Psi[AW]\|_{L^2}^2 = |\langle AW, \Psi[AW]\rangle| \lesssim \|A\||\langle AW, \Psi\rangle|,
$$

so it only remains to control the factors involving Φ. Testing (95) with $-\frac{1}{g}\Phi$ and using (96), we obtain:

$$
\begin{aligned}
\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2 &\lesssim -\Re\langle W, \frac{1}{g}\Phi\rangle = \Re\langle (1-(g(\frac{\partial_y}{k} - it))^2)\Phi, \frac{1}{g}\Phi\rangle \\
&\lesssim \|\frac{1}{g}\|_{W^{1,\infty}}\|\Psi\|_{\tilde{H}^1}\|\Phi\|_{\tilde{H}^1}.
\end{aligned}
$$

Dividing by $\|\Phi\|_{\tilde{H}^1} := \sqrt{\|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - ikt)\Phi\|_{L^2}^2}$, then provides the desired estimate. $\square$

This concludes our proof of Theorem 4.13 and thus establishes L^2 stability for a large class of strictly monotone shear flows in a finite periodic channel. Unlike the setting of an infinite periodic channel, where in Section 2.3 the L^2 stability results could be extended to arbitrarily high Sobolev norms, in the following subsections we show that boundary effects introduce additional correction terms (even in the constant coefficient model), which qualitatively change the stability behavior of the equations.

3.2. H^1 **stability.** In order to extend the stability results to H^1, we proceed as in Section 2.3 and differentiate the linearized Euler equations for a finite periodic channel, (91). We note, that $\partial_y \Psi$ and $\partial_y \Phi$ do not anymore satisfy zero Dirichlet boundary conditions, and thus split $\partial_y \Phi = \Phi^{(1)} + H^{(1)}$:

$$
\begin{aligned}
\partial_t \partial_y W &= \frac{if}{k}(\Phi^{(1)} + H^{(1)}) + \frac{if'}{k}\Phi, \\
(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(1)} &= \partial_y W + [(g(\partial_y - it))^2, \partial_y]\Phi, \\
\Phi^{(1)}_{y=0,1} &= 0, \\
H^{(1)} &= \partial_y \Phi - \Phi^{(1)}.
\end{aligned} \tag{97}
$$

The *homogeneous correction*, $H^{(1)}$, hence satisfies

$$
\begin{aligned}
(-1 + (g(\frac{\partial_y}{k} - it))^2)H^{(1)} &= 0, \\
H^{(1)}|_{y=0,1} &= \partial_y \Phi_{y=0,1}.
\end{aligned} \tag{98}
$$

The control of the contributions by Φ and $\Phi^{(1)}$ is obtained as in Section 3.1, while the control of the boundary corrections due to $H^{(1)}$ is given by the following lemmata.

LEMMA 4.7 (H^1 boundary contributions). *Let $A(t)$ be a diagonal operator comparable to the identity, i.e.*

$$
\begin{aligned}
A(t) &: e^{iny} \mapsto A_n(t) e^{iny}, \\
1 &\lesssim A_n(t) \lesssim 1,
\end{aligned}
$$

$\omega_0 \in H^1, f, g \in W^{2,\infty}$ *and suppose that* $0 < c < g < c^{-1} < \infty$. *Let further W be the solution of* (97).

Then, for any $0 < \gamma, \beta < \frac{1}{2}$ *there exists a constant* $C = C(\gamma, \beta, \|f\|_{W^{2,\infty}}, c, \|g\|_{W^{2,\infty}})$, *such that*

$$
|\langle A\partial_y W, f H^{(1)} \rangle| \leq C < t >^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2 + C \sum_n < t >^{-2\gamma} < \frac{n}{k} - t >^{-2\beta} |(\partial_y W)_n|^2.
$$

If additionally $\omega_0|_{y=0,1} \equiv 0$, *then for any* $0 < \beta < \frac{1}{2}$ *there exists a constant* $C = C(\gamma, \beta, \|f\|_{W^{2,\infty}}, c, \|g\|_{W^{2,\infty}})$, *such that*

$$
|\langle A\partial_y W, f H^{(1)} \rangle| \lesssim \sum_n < t >^{-1} < \frac{n}{k} - t >^{-2\beta} |(\partial_y W)_n|^2.
$$

LEMMA 4.8 (H^1 stream function estimate). *Let A, W, f, g satisfy the assumptions of Lemma 4.7. Then*

$$
|\langle A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi \rangle| \lesssim k^{-1}\|f\|_{W^{2,\infty}} (\|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2).
$$

Using Lemmata 4.7, 4.8 and the lemmata of Section 3.1, we prove H^1 stability.

THEOREM 4.14 (H^1 stability for the finite periodic channel). *Let W be a solution of the linearized Euler equations,* (97)*, and suppose that $f, g \in W^{2,\infty}$ and that there exists $c > 0$, such that*

$$0 < c < g < c^{-1} < \infty.$$

Further define a diagonal weight $A(t)$:

$$(99)\qquad \begin{aligned} &A(t) : e^{iny} \mapsto A_n(t)e^{iny}, \\ &A_n(t) = \exp\left(-\int_0^t < \frac{n}{k} - \tau >^{-2} + < \tau >^{-2\gamma} < \frac{n}{k} - \tau >^{-2\beta} \, d\tau\right), \end{aligned}$$

where $\beta, \gamma < \frac{1}{2}$ and $2\gamma + 2\beta > 1$. Also suppose that

$$\|f\|_{W^{2,\infty}} L$$

is sufficiently small. Then, for any $\omega_0 \in H^1([0,1])$, the solution W of (91) *(and hence* (97)*) with initial datum ω_0, satisfies*

$$\|W(t)\|_{H^1}^2 \lesssim I(t) := \langle A(t)W, W\rangle + \langle A(t)\partial_y W, \partial_y W \lesssim I(0) \lesssim \|\omega_0\|_{H^1}.$$

If additionally $\omega_0|_{y=0,1} \equiv 0$, then $I(t)$ is non-increasing.

PROOF OF THEOREM 4.14. Let W be a solution of (97), then we compute

$$\frac{d}{dt}\langle \partial_y W, A\partial_y W\rangle = \langle \dot{A}\partial_y W, \partial_y W\rangle + 2\Re\langle A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi\rangle + 2\Re\langle A\partial_y W, \frac{if}{k}H^{(1)}\rangle.$$

Using Lemma 4.8 in combination with Lemma 4.4, we estimate the second term by:

$$\begin{aligned} &2\Re\langle A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi\rangle \\ \lesssim\ &\frac{\|f\|_{W^{2,\infty}}}{|k|}(\|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2) \\ \lesssim\ &\frac{\|f\|_{W^{2,\infty}}}{|k|}(|\langle W, \dot{A}W\rangle| + |\langle \partial_y W, \dot{A}\partial_y W\rangle|). \end{aligned}$$

Using Lemma 4.7, the last term is controlled by:

$$\begin{aligned} &2\Re\langle A\partial_y W, \frac{if}{k}H^{(1)}\rangle \\ \lesssim\ &\frac{\|f\|_{W^{2,\infty}}}{|k|} < t >^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2 + \frac{\|f\|_{W^{2,\infty}}}{|k|}\sum_n < t >^{-2\gamma} < \frac{n}{k} - t >^{-2\beta} |(\partial_y W)_n|^2, \end{aligned}$$

or by

$$C_1 \frac{\|f\|_{W^{2,\infty}}}{|k|}\sum_n < t >^{-1} < \frac{n}{k} - t >^{-2\beta} |(\partial_y W)_n|^2,$$

if $\omega_0|_{y=0,1} \equiv 0$.

Hence, for

$$\sup_{k\neq 0} \frac{\|f\|_{W^{2,\infty}}}{|k|}$$

sufficiently small,

$$2\Re\langle A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi\rangle + 2\Re\langle A\partial_y W, \frac{if}{k}H^{(1)}\rangle$$

can be absorbed by

$$\langle \dot{A}\partial_y W, \partial_y W\rangle = -\sum_n A_n(t)\left(< \frac{n}{k} - t >^{-2} + < t >^{-2\gamma} < \frac{n}{k} - t >^{-2\beta}\right)|(\partial_y W)_n|^2 \leq 0.$$

Thus, $I(t)$ satisfies

$$\frac{d}{dt}I(t) \lesssim < t >^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2,$$

or, in the case of vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$,

$$\frac{d}{dt}I(t) \leq 0.$$

Integrating these inequalities in time concludes the proof. $\square$

Proof of Lemma 4.7. Similar to the construction of Lemma 4.3, let u_j, $j = 1, 2$, be solutions of

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)u = 0$$

with boundary values

$$\begin{aligned} u_1(0) = u_2(1) = 1, \\ u_1(1) = u_2(0) = 0. \end{aligned} \tag{100}$$

Recalling the sequence of transformations turning ϕ into Φ, the functions u_j are given by linear combinations of the homogeneous solutions

$$e^{\pm kG(y)+ikty},$$

where $G(y) = U^{-1}(y)$ satisfies $G(y)' = g(y)$.

Further recalling the boundary conditions in (98), $H^{(1)}$ is hence given by

$$H^{(1)} = \partial_y\Phi(0)u_1 + \partial_y\Phi(1)u_2.$$

In order to compute $\partial_y\Phi|_{y=0,1}$, we test the equation for Φ in (91), i.e.

$$\begin{aligned} (-1 + (g(y)(\frac{\partial_y}{k} - it))^2)\Phi = W, \\ \Phi|_{y=0,1} = 0, \end{aligned} \tag{101}$$

with u_j:

$$\begin{aligned} \langle W, u_j\rangle &= \langle(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi, u_j\rangle \\ &= u_j g(\frac{\partial_y}{k} - it)(g\Phi)\Big|_{y=0}^{1} - g\Phi(\frac{\partial_y}{k} - it)(gu_j)\Big|_{y=0}^{1} \\ &\quad + \langle\Phi, (-1 + (g(\frac{\partial_y}{k} - it))^2)u_j\rangle \\ &= u_j\frac{g^2}{k}\partial_y\Phi\Big|_{y=0}^{1}, \end{aligned}$$

where we used that $\Phi|_{y=0,1} = 0$. Using the boundary values of u_j, (100),

$$\begin{aligned} u_1\frac{g^2}{k}\partial_y\Phi\Big|_{y=0}^{1} &= -\frac{g^2(0)}{k}\partial_y\Phi|_{y=0}, \\ u_2\frac{g^2}{k}\partial_y\Phi\Big|_{y=0}^{1} &= \frac{g^2(1)}{k}\partial_y\Phi|_{y=1}. \end{aligned}$$

As $k \neq 0$ and $g^2 > c > 0$, we may solve for $\partial_y\Phi|_{y=0,1}$:

$$H^{(1)} = \frac{k}{g^2(0)}\langle W, u_1\rangle u_1 - \frac{k}{g^2(1)}\langle W, u_2\rangle u_2. \tag{102}$$

The boundary contribution can thus be explicitly computed in terms of u_1, u_2:

$$\langle A\partial_y W, fH^{(1)}\rangle = \frac{k}{g^2(0)}\langle W, u_1\rangle\langle A\partial_y W, fu_1\rangle - \frac{k}{g^2(1)}\langle W, u_2\rangle\langle A\partial_y W, fu_2\rangle.$$

As the homogeneous solutions $e^{\pm kG(y)+ikty}$ and thus u_1, u_2 are highly oscillatory, we integrate $k\langle W, u_j\rangle$ by parts and use that the evolution of (90) preserves boundary values, i.e. $W|_{y=0,1} = \omega_0|_{y=0,1}$. Denoting primitive functions of u_j by U_j and using that

$$e^{\pm kG(y)+ikty} = \frac{1}{\pm kg + ikt}\partial_y e^{\pm kG(y)+ikty},$$

we therefore obtain

$$\begin{aligned} k\langle W, u_j\rangle &= kU_j\omega_0|_{y=0}^1 - \langle \partial_y W, kU_j\rangle \\ &\leq \mathcal{O}(t^{-1})(\|\omega_0\|_{H^1} + |\langle \partial_y W, u_1\rangle| + |\langle \partial_y W, u_2\rangle|). \end{aligned} \tag{103}$$

Using Young's inequality, this yields a bound by

$$\begin{aligned} \left|\langle A\partial_y W, fH^{(1)}\rangle\right| &\lesssim < t >^{-1} (|\langle \partial_y W, u_1\rangle|^2 + |\langle A\partial_y W, fu_j\rangle|^2) \\ &\quad + < t >^{-1} |\langle A\partial_y W, fu_j\rangle| \|\omega_0\|_{H^1} \\ &\lesssim < t >^{-2\gamma} (|\langle \partial_y W, u_1\rangle|^2 + |\langle A\partial_y W, fu_j\rangle|^2) \\ &\quad + < t >^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2, \end{aligned} \tag{104}$$

where $0 < \gamma < \frac{1}{2}$ is chosen close to $\frac{1}{2}$.

Expanding $\partial_y W$ in our basis and choosing $0 < \beta < \frac{1}{2}$ close to $\frac{1}{2}$, we further estimate

$$\begin{aligned} |\langle \partial_y W, u_j\rangle| &\lesssim \sum_n |(\partial_y W)_n||\langle e^{iny}, u_j\rangle| \lesssim \sum_n |(\partial_y W)_n|\frac{1}{|k + i(n-kt)|} \\ &\leq \frac{1}{k}\|(\partial_y W)_n < \frac{n}{k} - t >^{-\beta} \|_{l_n^2} \| < \frac{n}{k} - t >^{-1+\beta} \|_{l_n^2} \\ &\lesssim_\beta \|(\partial_y W)_n < \frac{n}{k} - t >^{-\beta} \|_{l^2}. \end{aligned}$$

A similar bound also holds for $\langle A\partial_y W, fu_j\rangle$, where the constant further includes a factor $\|f\|_{W^{1,\infty}}$.

Thus, (104) can further be controlled by

$$\left|\langle A\partial_y W, fH^{(1)}\rangle\right| \lesssim < t >^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2 + \sum_n < t >^{-2\gamma} < \frac{n}{k} - t >^{-2\beta} |(\partial_y W)_n|^2.$$

The improved result for $\omega_0|_{y=0,1} \equiv 0$ similarly follows from (104), as in that case the term $< t >^{-1} |\langle A\partial_y W, fu_j\rangle|$ is not present. $\square$

Proof of Lemma 4.8. Using the vanishing boundary values of Φ and $\Phi^{(1)}$ and introducing

$$\begin{aligned} (-1 + (\frac{\partial_y}{k} - it)^2)\Psi[A\partial_y W] &= A\partial_y W, \\ \Psi[A\partial_y W]|_{y=0,1} &= 0, \end{aligned}$$

we integrate by parts to bound by

$$\begin{aligned} &\left|\left\langle (-1 + (\frac{\partial_y}{k} - it)^2)\Psi[A\partial_y W], \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi \right\rangle\right| \\ \leq &\left(\|\Psi\|_{L^2} + \|(\frac{\partial_y}{k} - it)\Psi\|_{L^2}\right)\frac{\|f\|_{W^{2,\infty}}}{k}\left(\|\Phi\|_{L^2} + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2} + \|\Phi^{(1)}\|_{L^2} + \|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{L^2}\right) \\ \leq &\frac{\|f\|_{W^{2,\infty}}}{k}(\|\Psi\|_{\tilde{H}^1}^2 + \|\Phi\|_{\tilde{H}^1}^2 + \|\Phi^{(1)}\|_{\tilde{H}^1}^2). \end{aligned}$$

In order to further estimate $\|\Phi^{(1)}\|_{\tilde{H}^1}$, we again use the vanishing boundary values of $\Phi^{(1)}$ and test

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(1)} = \partial_y W + [(g(\partial_y - it))^2, \partial_y]\Phi,$$
$$\Phi^{(1)}_{y=0,1} = 0,$$

with $-\frac{1}{g}\Phi^{(1)}$, to obtain that

$$\begin{aligned}\|\Phi^{(1)}\|^2_{\tilde{H}^1} \lesssim & - \langle (-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(1)}, \frac{1}{g}\Phi^{(1)} \rangle \\ \leq & - \langle (-1 + (\frac{\partial_y}{k} - it)^2)\Psi[\partial_y W], \frac{1}{g}\Phi^{(1)} \rangle + \langle [(g(\frac{\partial_y}{k} - it))^2, \partial_y]\Phi, \Phi^{(1)} \rangle \\ \lesssim & \|\Psi[\partial_y W]\|_{\tilde{H}^1}\|\Phi^{(1)}\|_{\tilde{H}^1} + \|\Phi\|_{\tilde{H}^1}\|\Phi^{(1)}\|_{\tilde{H}^1}.\end{aligned}$$

Using this inequality and Lemma 4.6 to estimate $\|\Phi\|_{\tilde{H}^1} \lesssim \|\Psi\|_{\tilde{H}^1}$, then concludes the proof. $\square$

As a consequence of the H^1 stability result, Theorem 4.14, Theorem 4.4 of Section 1 yields damping with rate t^{-1}, i.e.

$$\begin{aligned}\|v - \langle v \rangle_x\|_{L^2(\mathbb{T}_L \times [0,1])} \leq \mathcal{O}(t^{-1})\|W(t)\|_{L^2(\mathbb{T}_L \times [0,1])} &\leq \mathcal{O}(t^{-1})\|\omega_0\|_{L^2(\mathbb{T}_L \times [0,1])}, \\ \|v_2\|_{L^2(\mathbb{T}_L \times [0,1])} &\leq \mathcal{O}(t^{-1})\|\omega_0\|_{L^2(\mathbb{T}_L \times [0,1])}.\end{aligned}$$

As discussed in Section 3 of Chapter 2, the first estimate thus already attains the optimal rate. The estimate for v_2, however, does not yet provide an integrable decay rate, $\mathcal{O}(t^{-1-\epsilon})$, and thus, in particular, is not sufficient to prove scattering.

In the following section, we thus prove H^2 stability and hence linear inviscid damping with the optimal rates as well as scattering. There, we additionally require our perturbations to satisfy zero Dirichlet boundary conditions, $\omega_0|_{y=0,1} = 0$.

As we discuss in Section 6 and Chapter 5, this is not only a technical restriction: We show that otherwise $\partial_y W$ asymptotically develops a logarithmic singularity at the boundary, which by the trace theorem in particular forbids stability in any Sobolev space more regular than $H^{\frac{3}{2}}_y$. Furthermore, this restriction is shown to be optimal in the sense that stability holds in all subcritical (periodic) fractional Sobolev spaces $H^s_y(\mathbb{T}), s < \frac{3}{2}$, and blow up occurs in all supercritical Sobolev spaces, $H^s_y([0,1]), s > \frac{3}{2}$.

3.3. H^2 **stability.** Following a similar approach as in the previous Subsection 3.2, we obtain H^2 stability and hence linear inviscid damping with the optimal rates and scattering for a large class of monotone shear flows in a finite periodic channel. As we discuss in Section 6 and Chapter 5, for this stability result it is necessary to restrict to perturbations with zero Dirichlet data, $\omega_0|_{y=0,1} = 0$.

We again differentiate our equation and introduce homogeneous correction terms $H^{(1)}, H^{(2)}$. Let thus W be a solution of (91), then $\partial_y^2 W$ satisfies

$$\begin{aligned}\partial_t \partial_y^2 W &= \frac{if}{k}(\Phi^{(2)} + H^{(2)}) + \frac{2f'}{ik}(\Phi^{(1)} + H^{(1)}) + \frac{f''}{ik}\Phi, \\ (-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(2)} &= \partial_y^2 W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi, \\ \Phi^{(2)}_{y=0,1} &= 0.\end{aligned} \tag{105}$$

Here the *homogeneous correction* $H^{(2)}$ satisfies

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)H^{(2)} = 0,$$
$$H^{(2)}|_{y=0,1} = \partial_y^2 \Phi|_{y=0,1}.$$

We recall that the equations satisfied by $\partial_y W, \Phi^{(1)}, H^{(1)}$ are given by (97) and (98), respectively.

As in Section 3.2, we introduce several lemmata to control boundary corrections. Using these lemmata, we then prove the main stability result, Theorem 4.15.

LEMMA 4.9 (H^2 boundary contribution I). *Let $A(t)$ be a diagonal operator comparable to the identity, i.e.*

$$A : e^{iny} \mapsto A_n e^{iny},$$
$$1 \lesssim A_n \lesssim 1,$$

and let W be a solution of (105) *with initial datum $\omega_0 \in H^2([0,1])$ with $\omega_0|_{y=0,1} = 0$. Suppose further that $f, g \in W^{3,\infty}$ and k (or L respectively) satisfy the assumptions of the H^1 stability result, Theorem 4.14. Then $H^{(1)}$ satisfies*

$$\|H^{(1)}\|_{\tilde{H}^1}^2 \lesssim < t >^{-2} \|W\|_{H^1}^2 \lesssim < t >^{-2} \|\omega_0\|_{H^1}^2$$

and for any $0 < \beta, \gamma < \frac{1}{2}$,

$$|\langle A\partial_y^2 W, \frac{if'}{k} H^{(1)}\rangle| \lesssim_{\beta,\gamma} \|f\|_{W^{2,\infty}} k^{-1} \Big(\log^2(t) < t >^{-2(1-\gamma)} \|\omega_0\|_{H^2}^2$$
$$+ \sum_n < t >^{-2\gamma} < \frac{n}{k} - t >^{-2\beta} |(A\partial_y^2 W)_n|^2 \Big).$$

LEMMA 4.10 (H^2 boundary contribution II). *Let A, f, g, W, k as in Lemma 4.9. Then for $0 < \gamma, \beta < \frac{1}{2}$ there exits a constant $C = C(f, g, k, \beta, \gamma)$, such that*

$$|\langle A\partial_y^2 W, \frac{if}{k} H^{(2)}\rangle| \leq C \log^2(t) < t >^{-2(1-\gamma)} \|\omega_0\|_{H^2}^2$$
$$+ C \sum_n < t >^{-2\gamma} < \frac{n}{k} - t >^{-2\beta} |(\partial_y^2 W)_n|^2.$$

LEMMA 4.11 (H^2 stream function estimate I). *Let A, f, g, W, k as in Lemma 4.9. Then,*

$$|\langle A\partial_y W, \frac{if}{k} \Phi^{(2)}\rangle| \lesssim k^{-1} \|f\|_{W^{1,\infty}} (\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2).$$

LEMMA 4.12 (H^2 stream function estimate II). *Let A, f, g, k, W as in Lemma 4.9. Then,*

$$| < A\partial_y W, \frac{if}{k} \Phi^{(1)} + \frac{if'}{k} \Phi > | \lesssim \frac{1}{|k|} \|f\|_{W^{2,\infty}} (\|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2).$$

THEOREM 4.15 (H^2 stability for the finite periodic channel). *Let f, g, W, k as in Lemma 4.9 and let $A(t)$ be defined as in Theorem 4.14, i.e. let $A(t)$ be a diagonal weight:*

$$A(t) : e^{iny} \mapsto A_n(t) e^{iny},$$
$$A_n(t) = \exp\left(- \int_0^t < \frac{n}{k} - \tau >^{-2} + < \tau >^{-2\gamma} < \frac{n}{k} - \tau >^{-2\beta} d\tau \right), \tag{106}$$

where $\beta, \gamma < \frac{1}{2}$ and $2\gamma + 2\beta > 1$. Further suppose that

$$\|f\|_{W^{3,\infty}} L$$

is sufficiently small. Then, for any $\omega_0 \in H^2([0,1])$ with vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$,

$$E_2(t) := \langle A(t)W, W\rangle + \langle A(t)\partial_y W, \partial_y W\rangle + \langle A(t)\partial_y^2 W, \partial_y^2 W\rangle$$

satisfies

$$\|W(t)\|_{H^2} \lesssim E_2(t) \lesssim E_2(0) \lesssim \|\omega_0\|_{H^2}.$$

Proof of Theorem 4.15. The control of

$$\langle A(t)W, W\rangle + \langle A(t)\partial_y W, \partial_y W\rangle$$

has been established in Theorem 4.14.

Differentiating $\langle A(t)\partial_y^2 W, \partial_y^2 W\rangle$ in time, we have to control

$$\langle A\partial_y^2 W, \frac{if}{k}\Phi^{(2)} + \frac{2f'}{ik}\Phi^{(1)} + \frac{f''}{ik}\Phi\rangle + \langle A\partial_y^2 W, \frac{if}{k}H^{(2)} + \frac{2f'}{ik}H^{(1)}\rangle.$$

As $\Phi^{(2)}, \Phi^{(1)}$ and Φ have zero boundary values, we integrate

$$(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[A\partial_y^2 W] = A\partial_y^2 W,$$
$$\Psi[A\partial_y^2 W]_{y=0,1} = 0,$$

by parts and bound by the $\tilde{H}^1$ norm:

$$\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1} \frac{\|f\|_{W^{3,\infty}}}{k} \left(\|\Phi^{(2)}\|_{\tilde{H}^1} + \|\Phi^{(1)}\|_{\tilde{H}^1} + \|\Phi\|_{\tilde{H}^1}\right).$$

Lemmata 4.11 and 4.12 provide control by

$$\frac{1}{|k|}\|f\|_{W^{3,\infty}}(\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1}^2 + \|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[\partial_y W]\|_{\tilde{H}^1}^2 + \|\Psi[W]\|_{\tilde{H}^1}^2). \tag{107}$$

Supposing that

$$\sup \frac{1}{|k|}\|f\|_{W^{3,\infty}} \tag{108}$$

is sufficiently small, and using Lemma 4.5, (107) can be absorbed by

$$\langle W, \dot{A}W\rangle + \langle \partial_y W, \dot{A}\partial_y W\rangle + \langle \partial_y^2 W, \dot{A}\partial_y^2 W\rangle. \tag{109}$$

Using Lemmata 4.9 and 4.10 and supposing again that (108) is sufficiently small, the boundary contributions

$$\langle A\partial_y^2 W, \frac{if}{k}H^{(2)} + \frac{2f'}{ik}H^{(1)}\rangle,$$

can be partially absorbed in (109), with the remaining terms estimated by

(110)
$$< t >^{-2} \|\omega_0\|_{H^1}^2 + \|f\|_{W^{2,\infty}}|\frac{1}{k}| < t >^{-2(1-\gamma)} \|\omega_0\|_{H^2}^2 + \log^2(t) < t >^{-2(1-\gamma)} \|\omega_0\|_{H^2}^2.$$

We thus obtain that $E_2(t)$ satisfies

$$\begin{aligned}\partial_t E_2(t) &\leq \frac{d}{dt}\left(\langle A(t)W, W\rangle + \langle A(t)\partial_y W, \partial_y W\rangle + \langle A(t)\partial_y^2 W, \partial_y^2 W\rangle\right)\\ &\lesssim (< t >^{-2} + \log^2(t) < t >^{-2(1-\gamma)})\|\omega_0\|_{H^2}^2.\end{aligned}$$

As $0 < \gamma < \frac{1}{2}$, this is integrable and thus yields the result. □

It remains to prove the Lemmata 4.9-4.12.

Proof of Lemma 4.9. We recall from the proof of Lemma 4.7 that $H^{(1)}$ is explicitly given by

$$H^{(1)} = \partial_y\Phi(0)u_1 + \partial_y\Phi(1)u_2.$$

By the triangle inequality, we thus estimate by

$$\|H^{(1)}\|_{\tilde{H}^1} \lesssim |\partial_y\Phi(t,0)|\|u_1(t)\|_{\tilde{H}^1} + |\partial_y\Phi(t,1)|\|u_1(t)\|_{\tilde{H}^1}.$$

We further recall that the homogeneous solutions u_1, u_2 are of the form

$$a(t)e^{kG(y)+ikty} + b(t)e^{kG(y)+ikty},$$

where $a(t), b(t)$ are chosen to satisfy the boundary conditions, (100). Hence, for any time t

$$\|u_j(t)\|_{\tilde{H}^1} \leq |a(t)|\|e^{kG(y)+ikty}\|_{\tilde{H}^1} + |b(t)|\|e^{-kG(y)+ikty}\|_{\tilde{H}^1} = |a(t)|\|e^{kG(y)}\|_{H^1} + |b(t)|\|e^{-kG(y)}\|_{H^1}.$$

Therefore, by direct computation of the coefficients a, b (analogously to Lemma 4.3),

$$\|u_j(t)\|_{\tilde{H}^1} < C < \infty,$$

uniformly in time.

Thus, $H^{(1)}$ satisfies

$$\|H^{(1)}\|_{\tilde{H}^1} \lesssim |\partial_y\Phi(t,0)| + |\partial_y\Phi(t,1)|.$$

Recalling the explicit characterization of $\partial_y\Phi|_{y=0,1}$, (102),

$$\partial_y\Phi|_{y=0} = \frac{k}{g^2(0)}\langle W, u_1\rangle,$$
$$\partial_y\Phi|_{y=1} = -\frac{k}{g^2(1)}\langle W, u_2\rangle,$$

and the subsequent estimate, (103),

$$|k\langle W, u_j\rangle| \lesssim <t>^{-1} (\|\omega_0\|_{H^1} + |\langle\partial_y W, u_1\rangle| + |\langle\partial_y W, u_2\rangle|),$$

we further control

$$\|H^{(1)}\|_{\tilde{H}^1} \lesssim <t>^{-1} \|W\|_{H^1}, \tag{111}$$

which yields the first result.

In order to estimate

$$\begin{aligned}\langle A\partial_y^2 W, \frac{if'}{k}H^{(1)}\rangle &= \partial_y\Psi(0)\langle A\partial_y^2 W, u_1\rangle + \partial_y\Psi(1)\langle A\partial_y^2 W, u_2\rangle \\ &\lesssim <t>^{-1} |\langle A\partial_y^2 W, u_j\rangle|,\end{aligned}$$

we proceed as in Lemma 4.7 and expand $\langle A\partial_y^2 W, u_j\rangle$ in our basis. Thus, we obtain:

$$|\langle A\partial_y^2 W, u_j\rangle| \lesssim_\beta \|(A\partial_y^2 W)_n < \frac{n}{k} - t >^{-\beta} \|_{l^2},$$

for $0 < \beta < \frac{1}{2}$. Hence, by Young's inequality:

$$\begin{aligned}\langle A\partial_y^2 W, \frac{if'}{k}H^{(1)}\rangle &\lesssim <t>^{-1} \|W\|_{H^1}\|(A\partial_y^2 W)_n < \frac{n}{k} - t >^{-\beta} \|_{l^2} \\ &\lesssim <t>^{-2(1-\gamma)} \|W\|_{H^1}^2 + \sum_n <t>^{-2\gamma} < \frac{n}{k} - t >^{-2\beta} |(A\partial_y^2 W)_n|^2.\end{aligned}$$

□

Proof of Lemma 4.10. We follow the same strategy as in the proof of Lemma 4.7 and Lemma 4.9 and explicitly compute

$$
\begin{aligned}
H^{(2)} &= \partial_y^2\Phi(0)u_1 + \partial_y^2\Phi(1)u_2, \\
\langle A\partial_y^2 W, \frac{if}{k}H^{(2)}\rangle &= \partial_y^2\Phi(0)\langle A\partial_y^2 W, \frac{if}{k}u_1\rangle + \partial_y^2\Phi(1)\langle A\partial_y^2 W, \frac{if}{k}u_2\rangle \\
&\lesssim |\partial_y^2\Phi|_{y=0,1}| \left\| < \frac{n}{k} - t >^{-\beta} (A\partial_y^2 W)_n \right\|_{l_n^2}.
\end{aligned}
\tag{112}
$$

It hence remains to estimate $\partial_y^2\Phi|_{y=0,1}$. We thus expand the equation for the stream function Φ in the linearized Euler equations, (91)),

$$
(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi = W,
$$

and obtain

$$
-\Phi + g^2k^{-2}\partial_y^2\Phi + k^{-1}gg'(\frac{\partial_y}{k} - it)\Phi - g^2 it\frac{\partial_y}{k}\Phi + g^2t^2\Phi = W.
$$

Thus, using that Φ and W vanish at the boundary, $\partial_y^2\Phi|_{y=0,1}$ satisfies

$$
g^2\partial_y^2\Phi|_{y=0,1} = (-gg' + iktg^2)\partial_y\Phi|_{y=0,1}.
$$

Dividing by g^2 (which we required to be bounded away from zero), we may thus solve for $\partial_y^2\Phi_{y=0,1}$:

$$
\partial_y^2\Phi|_{y=0,1} = \frac{-gg' + iktg^2}{g^2}\partial_y\Phi|_{y=0,\psi} = \mathcal{O}(kt)\partial_y\Phi|_{y=0,1}.
$$

Again recalling the explicit characterization of $\partial_y\Phi|_{y=0,1}$, (102),

$$
\begin{aligned}
\partial_y\Phi|_{y=0} &= \frac{k}{g^2(0)}\langle W, u_1\rangle, \\
\partial_y\Phi|_{y=1} &= -\frac{k}{g^2(1)}\langle W, u_2\rangle,
\end{aligned}
$$

from the proof of Lemma 4.7, we further compute

$$
\begin{aligned}
\mathcal{O}(kt)\partial_y\Phi|_{y=0,1} &\lesssim k^2t\langle W, u_j\rangle \lesssim k\langle\partial_y W, u_j\rangle + ku_jW|_{y=0}^1 \\
&\lesssim < t >^{-1} \langle\partial_y^2 W, u_j\rangle + < t >^{-1} \partial_y W u_j|_{y=0}^1 + ku_jW|_{y=0}^1 \\
&= < t >^{-1} \langle\partial_y^2 W, u_j\rangle + < t >^{-1} \partial_y W u_j|_{y=0}^1,
\end{aligned}
$$

where we again used that $W|_{y=0,1} = \omega_0|_{y=0,1} \equiv 0$. The first term can again be estimated by

$$
< t >^{-1} \| < \frac{n}{k} - t >^{-\beta} (A\partial_y^2 W)_n\|_{l_n^2}
$$

and thus yields a contribution of the desired form.

To estimate the second term,

$$
< t >^{-1} \partial_y W u_j|_{y=0}^1, \tag{113}
$$

we restrict the evolution equation for $\partial_y W$, (97), to the boundary and obtain

$$
\partial_t\partial_y W|_{y=0,1} = \frac{f'}{ik}\Phi|_{y=0,1} + \frac{f}{ik}\partial_y\Phi_{y=0,1} = \frac{f}{ik}\partial_y\Phi \lesssim |\langle W, u_j\rangle|.
$$

Controlling the right-hand-side by $\mathcal{O}(t^{-1})\|W\|_{H^1}$ and using the H^1 stability result, Theorem 4.14, we thus obtain a logarithmic control

$$
|\partial_y W|_{y=0,1}| = \mathcal{O}(\log(t))\|\omega_0\|_{H^1}.
$$

Hence, (113) can be bounded by

$$
< t >^{-1} \partial_y W u_j|_{y=0}^1 \lesssim \log(t) < t >^{-1} \|\omega_0\|_{H^1}.
$$

Using these estimates, we may further estimate equation (112) by

$$\begin{aligned}\langle A\partial_y^2 W, \frac{if}{k} H^{(2)}\rangle \lesssim & (< t >^{-1} \| < \frac{n}{k} - t >^{-\beta} (A\partial_y^2 W)_n \|_{l^2_n} + < t >^{-1} \log(t) \|\omega_0\|_{H^1}) \\ & \cdot \| < \frac{n}{k} - t >^{-\beta} (A\partial_y^2 W)_n \|_{l^2_n} \\ \lesssim & \log^2(t) < t >^{-2(1-\gamma)} \|\omega_0\|_{H^1}^2 + \sum_n < t >^{-2\gamma} < \frac{n}{k} - t >^{-2\beta} |(A\partial_y^2 W)_n|^2 \\ & + \sum_n < t >^{-1} < \frac{n}{k} - t >^{-2\beta} |(A\partial_y^2 W)_n|^2.\end{aligned}$$

□

Proof of Lemma 4.11. We introduce

$$\begin{aligned}(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[A\partial_y^2 W] &= A\partial_y^2 W, \\ \Psi[A\partial_y^2 W]_{y=0,1} &= 0,\end{aligned}$$

and use the vanishing boundary values of $\Phi^{(2)}$ to integrate by parts and obtain

$$|\langle A\partial_y W, \frac{if}{k}\Phi^{(2)}\rangle| \lesssim k^{-1}\|f\|_{W^{1,\infty}} (\|\Psi[A\partial_y^2 W]\|_{\tilde{H}^1}^2 + \|\Phi^{(2)}\|_{\tilde{H}^1}^2).$$

It thus remains to control $\|\Phi^{(2)}\|_{\tilde{H}^1}^2$. Testing

$$\begin{aligned}(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(2)} &= \partial_y^2 W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi, \\ \Phi^{(2)}_{y=0,1} &= 0,\end{aligned}$$

with $-\frac{1}{g}\Phi^{(2)}$, we estimate

$$\begin{aligned}\|\Phi^{(2)}\|_{\tilde{H}^1}^2 \lesssim & - \langle (-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(2)}, \frac{1}{g}\Phi^{(2)}\rangle \\ \lesssim & \|\Psi[\partial_y^2 W]\|_{\tilde{H}^1} \|\Phi^{(2)}\|_{\tilde{H}^1} + \langle [(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi, \frac{1}{g}\Phi^{(2)}\rangle \\ \lesssim & \|\Phi^{(2)}\|_{\tilde{H}^1} (\|\Psi[\partial_y^2 W]\|_{\tilde{H}^1} + \|\partial_y \Phi\|_{\tilde{H}^1} + \|\Phi\|_{\tilde{H}^1}).\end{aligned}$$

Using the triangle inequality

$$\|\partial_y \Phi\|_{\tilde{H}^1} \lesssim \|\Phi^{(1)}\|_{\tilde{H}^1} + \|H^{(1)}\|_{\tilde{H}^1},$$

Lemma 4.9 and Lemma 4.8 then provide the desired control. □

Proof of Lemma 4.12. We introduce

$$\begin{aligned}(-1 + (\frac{\partial_y}{k} - it)^2)\Psi[A\partial_y^2 W] &= \partial_y^2 W, \\ \Psi[A\partial_y^2 W]_{y=0,1} &= 0,\end{aligned}$$

and use the vanishing boundary values of $\Phi^{(1)}$ and Φ to integrate by parts, to obtain

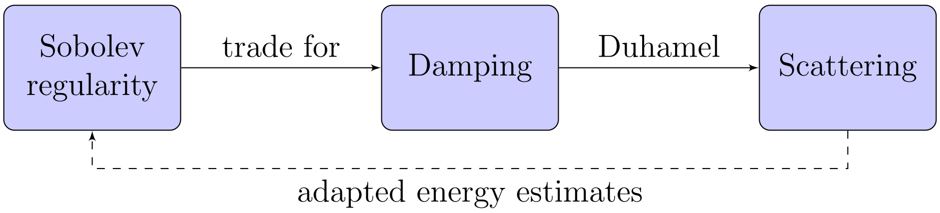
$$| < A\partial_y W, \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi > | \lesssim |k^{-1}| \|f\|_{W^{2,\infty}} (\|\Psi[A\partial_y W]\|_{\tilde{H}^1}^2 + \|\Phi^{(1)}\|_{\tilde{H}^1}^2 + \|\Phi[W]\|_{\tilde{H}^1}^2).$$

Lemma 4.8 then provides the desired control. □

With these stability results, we now have the desired control on $\|W\|_{H^2}$ and hence, as we discuss in the following section, can prove linear inviscid damping with the optimal algebraic decay rates for a large class of strictly monotone shear flows in a finite periodic channel.

4. Scattering and consistency

In this section, we combine the results of the previous sections and thus close our strategy to prove linear inviscid damping for monotone shear flows:



Theorem 4.16 (Linear inviscid damping for the infinite periodic channel and finite periodic channel). *Let* $\omega_0 \in L^2_x H^2_y$ *with* $\langle \omega_0 \rangle_x \equiv 0$ *and let* W *solve*

$$\begin{aligned} \partial_t W &= f \partial_x \Phi, \\ (\partial_x^2 + (g(\partial_y - t\partial_x))^2)\Phi &= W, \end{aligned} \tag{114}$$

either on the infinite periodic channel, $\mathbb{T}_L \times \mathbb{R}$*, or finite periodic channel,* $\mathbb{T}_L \times [0,1]$*. Suppose that there exists* $c > 0$ *such that*

$$\begin{aligned} c < g < c^{-1}, \\ \frac{1}{g}, f \in W^{3,\infty}, \end{aligned}$$

and that

$$L \|f\|_{W^{3,\infty}}$$

is sufficiently small. In the case of a finite periodic channel, additionally assume that

$$\omega_0(x,0) \equiv 0 \equiv \omega_0(x,1).$$

Then there exists a function $W_\infty \in L^2_x H^2_y$ *such that*

$$\begin{aligned} &\text{(Stability)} & \|W\|_{L^2_x H^2_y} &\lesssim \|\omega_0\|_{L^2_x H^2_y}, \\ &\text{(Damping)} & \|v - \langle v \rangle_x\|_{L^2} &= \mathcal{O}(t^{-1}), \\ & & \|v_2\|_{L^2} &= \mathcal{O}(t^{-2}), \\ &\text{(Scattering)} & W(t) &\to_{L^2} W_\infty, \end{aligned}$$

as $t \to \infty$*.*

Proof. Let $\omega_0 \in L^2_x H^2_y$ and f, g be given. Then by the stability results for the infinite channel, Theorem 4.11, and for the finite channel, Theorem 4.15, W satisfies

$$\|W\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y}.$$

As the mean in x is conserved, i.e.

$$\langle W(t) \rangle_x \equiv \langle \omega_0 \rangle_x \equiv 0,$$

we may apply Poincaré's theorem to deduce that

$$\|W\|_{H^{-1}_x H^2_y} \lesssim \|W\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y}.$$

The damping result, Theorem 4.4, of Section 1 then implies decay of the velocity field with the optimal algebraic rates.

Duhamel's formula in our scattering formulation is just integrating (114) in time and leads to:

$$W(t,x,y) = \omega_0(x, U^{-1}(y)) + \int_0^t f(y) V_2(\tau, x, y) d\tau, \tag{115}$$

where

$$V_2(t,x,y) = \partial_x \Phi(t,x,z) = v_2(t, x - ty, U^{-1}(y)).$$

Hence, as the change of variables $(x,y) \mapsto (x - tU(y), y)$ is an isometry and $y \mapsto U(y)$ is bilipschitz,

$$\|fV_2\|_{L^2} \leq \|f\|_{L^\infty} \|V_2\|_{L^2} \lesssim \|f\|_{L^\infty} \|v_2\|_{L^2} = \mathcal{O}(t^{-2}).$$

Thus, the integral in (115) is uniformly bounded in L^2 for all t and the improper integral for $t \to \pm\infty$ exists as a limit in L^2. Therefore,

$$W \xrightarrow{L^2} W_{\pm\infty} := \omega_0 + \lim_{t \to \pm\infty} \int_0^t f V_2(\tau) d\tau.$$

As $\|W\|_{L^2_x H^2_y} \lesssim \|\omega_0\|_{L^2_x H^2_y}$ uniformly in time, weak compactness and lower semicontinuity imply $W_{\pm\infty} \in L^2_x H^2_y$ and

$$\|W_{\pm\infty}\|_{H^2_y L^2_x} \lesssim \|w_0\|_{L^2_x H^2_y}.$$

□

Corollary 4.2 (L^2 scattering). *Let f, g, L be as in Theorem 4.16 and let $\omega_0 \in L^2$, then there exists $W_\infty \in L^2$ such that*

$$W(t,x,y) \xrightarrow{L^2} W_\infty, \text{ as } t \to \infty.$$

Proof. Let $H^2 \ni \omega_0^j \xrightarrow{L^2} \omega_0$ as $j \to \infty$. Then by the previous theorem there exist W^j_∞ such that

$$W^j(t) \xrightarrow{L^2} W^j_\infty, \text{ as } t \to \infty.$$

By the L^2 stability results, Theorem 4.10 of Section 2 and Theorem 4.13 of Section 3, and letting t tend to infinity, W^j_∞ is a Cauchy sequence in L^2. Denoting the limit by W_∞, a diagonal sequence argument yields $W(t) \xrightarrow{L^2} W_\infty$ as $t \to \infty$. □

A natural question following these linear inviscid damping and scattering results is, of course, whether such behavior also persists under the non-linear evolution. Bedrossian and Masmoudi, [**BM13b**], answer this question positively in the case of Couette flow, where the perturbations are required to be small in the Gevrey < 2 class to control nonlinear effects.

As a small step in the direction of similar results for monotone shear flows, we follow Bouchet and Morita, [**BM10**], and answer the simpler question of consistency. We recall from Section 1.6, that in the linearized Euler equations

$$\begin{aligned} \partial_t \omega + U(y) \partial_x \omega &= U'' \partial_x \phi, \\ \Delta \phi &= \omega, \end{aligned} \tag{116}$$

we neglected nonlinearity:

$$v \cdot \nabla \omega = v_1 \partial_x \omega + v_2 \partial_y \omega. \tag{117}$$

For a consistency result, we show that the nonlinearity, *when evolved with the linear dynamics*, is an integrable perturbation in the sense that

$$\sup_{T>0} \left\| \int_0^T v \cdot \nabla \omega dt \right\|_{L^2} < C < \infty.$$

In view of Theorem 4.16, at first sight we would expect decay of (117) with a rate of only $\mathcal{O}(t^{-1})$, as

$$\begin{aligned}
\|v_1\|_{L^2} &= \mathcal{O}(t^{-1}), \\
\|v_2\|_{L^2} &= \mathcal{O}(t^{-2}), \\
\|\partial_y \omega\|_{L^2} &= \|(\partial_y - tU'\partial_x)W\|_{L^2} = \mathcal{O}(t).
\end{aligned}$$

However, there is some additional cancellation, which can be used. In scattering coordinates $v \cdot \nabla\omega$ is given by

$$-(\partial_y - tU'\partial_x)\Phi\partial_x W + \partial_x\Phi(\partial_y - tU'\partial_x)W = \nabla^\perp\Phi \cdot \nabla W.$$

Combining the the stability results on ∇W and the damping results on $\nabla^\perp\Phi$ of Sections 2.3 and 3.2, we obtain quadratic decay

$$\|\nabla^\perp\Phi\|_{L^2} = \mathcal{O}(t^{-2}).$$

and thus consistency.

Lemma 4.13 (Consistency). *Let W be a solution to the linearized 2D Euler equation, (114), on $\mathbb{T} \times \mathbb{R}$ with initial datum $\omega_0 \in H^3_{x,y}(\mathbb{T}_L \times \mathbb{R})$. Suppose further that the assumptions of the Sobolev regularity result, Theorem 4.11, for $j = 3$, as well as of the damping result, Theorem 4.4, are satisfied. Then*

$$\|\nabla^\perp\Phi \cdot \nabla W\|_{L^2} = \mathcal{O}(t^{-2}).$$

In particular,

$$W(t) + \int^t \nabla^\perp\Phi(\tau)\nabla W(\tau)d\tau$$

is close to $W(t)$ in L^2 uniformly in time and there exist asymptotic profiles $W^{\pm\infty}_{con}$ such that

$$W(t) + \int^t \nabla^\perp\Phi(\tau)\nabla W(\tau)d\tau \xrightarrow{L^2} W^{\pm\infty}_{con}, \text{ as } t \to \pm\infty.$$

Proof. By Theorem 4.11, W satisfies

$$\|W(t)\|_{H^3_{x,y}} \lesssim \|\omega_0\|_{H^3_{x,y}}.$$

Hence, by Theorem 4.4,

$$\|\nabla^\perp\Phi\|_{L^2} = \mathcal{O}(t^{-2})\|W(t)\|_{H^3_{x,y}} = \mathcal{O}(t^{-2})\|\omega_0\|_{H^3_{x,y}}.$$

Using the Sobolev embedding,

$$\|\nabla W\|_{L^\infty_{x,y}} \lesssim \|W(t)\|_{H^3_{x,y}} \lesssim \|\omega_0\|_{H^3_{x,y}}.$$

An application of Hölder's inequality then gives the desired bound:

$$\|\nabla^\perp\Phi\nabla W\|_{L^2} \leq \|\nabla^\perp\Phi\|_{L^2}\|\nabla W\|_{L^\infty} = \mathcal{O}(t^{-2})\|\omega_0\|^2_{H^3_{x,y}}.$$

□

We remark that the regularity assumptions on ω_0 are not sharp. As we are, however, only interested in the qualitative property of consistency, we assume sufficiently much regularity to use a Sobolev embedding.

In the setting of a finite periodic channel $\mathbb{T} \times [0, 1]$, we thus far have only established stability in H^2, which is not sufficient for integrable decay of $\|\nabla^\perp\Phi\|_{L^2}$. Furthermore, in two dimensions $H^2_{x,y}$ regularity is critical for the Sobolev embedding. Hence, control of $\|W\|_{H^2(\mathbb{T}_L\times[0,1])}$ only yields $\nabla_{x,y}W \in \mathrm{BMO}(\mathbb{T}_L \times [0, 1])$ instead of $L^\infty(\mathbb{T}_L \times [0, 1])$.

A natural question is thus whether the stability result in a finite periodic channel can be improved to higher Sobolev spaces. As we sketch in Section 6, stability in H^3 is in general not possible.

Introducing fractional Sobolev spaces, in the following Chapter 5, we prove that the critical Sobolev space is given by $H^{5/2}$ in the sense that stability holds for any $H^s, s < 5/2$ and for any $s > 5/2$ the corresponding Sobolev norm will in general grow unboundedly as $t \to \infty$.

Based on these improvements, in Chapter 6 we further elaborate on consistency and additional implications of the stability and damping results.

The remainder of this chapter further discusses the choice of basis in the setting of a finite channel and analyzes the homogeneous corrections to the stream function in the case of Couette flow.

5. Bases and mapping properties

In this section, we elaborate on the role of boundary conditions, the choice of basis and the mapping properties of

$$\begin{aligned} W &\mapsto \langle W, \Psi \rangle, \\ (k^2 - (\partial_y - ikt)^2)\Psi &= W, \\ \Psi|_{y=0,1} &= 0. \end{aligned}$$

In analogy to the whole space setting, a first natural approach is via a Fourier basis, which we used in Section 3.1. There the coefficients of Ψ have been computed in Lemma 4.3:

LEMMA 4.14. *Let Ψ be as above, $n, m \in 2\pi\mathbb{Z}$, then*

$$\langle \Psi[e^{iny}], e^{imy} \rangle = \frac{\delta_{nm}}{k^2 + (n-kt)^2} + \frac{k}{(k^2 + (m-kt)^2)(k^2 + (n-kt)^2)}(a-b),$$

where a, b solve

$$\begin{pmatrix} e^{k+ikt} & e^{-k+ikt} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This choice of basis has a distinct advantage in its simplicity and good decoupling multiplier structure. In particular, we may easily prove Lemma 4.4 using Cauchy-Schwarz. We, however, see that we can not obtain a bounded map in $H^s, s \geq \frac{1}{2}$, in this way, as the decay is not fast enough and thus

$$\frac{n^s}{k^2 + (n-kt)^2} \not\in l^2.$$

When trying to use Schur's test instead, one encounters the problem of slow decay as $n, m \to \infty$ at an even earlier stage of our proof:

$$\begin{aligned} &\sup_m \sum_n < \frac{n}{k} - t >^\alpha < \frac{m}{k} - t >^\alpha |\langle \Psi[e^{iny}], e^{imy} \rangle| \\ &\leq 1 + k^{-3} \sum_n < \frac{n}{k} - t >^{\alpha-2} \lesssim_\alpha 1 + k^{-2}. \end{aligned} \tag{118}$$

Therefore, this approach does not even provide an l^2 estimate with optimal decay, but only a weaker variant of Lemma 4.4 with $\alpha < 1$.

Furthermore, testing against homogeneous solutions, we only obtain slow decay:

$$\langle e^{iny}, e^{\pm y+ity}\rangle = \frac{1}{\pm 1 + i(t-n)} e^{\pm y + i(t-n)y}|_{y=0}^{1} = \mathcal{O}(< n-t >^{-1}),$$

$$\langle \sin(ny), e^{\pm y+ity}\rangle = -n\frac{1}{\pm 1 + it}\langle \cos(ny), e^{\pm y+ity}\rangle = \mathcal{O}(n < t >^{-1} < n-t >^{-1}).$$

Considering a sin basis instead, we may make use of vanishing boundary terms to obtain additional cancellations and better coefficients:

LEMMA 4.15. *Let $n \in \pi\mathbb{N}$ and let $\Psi[\sin(ny)]$ be the solution of*

$$(k^2 - (\partial_y - ikt)^2)\Psi[\sin(ny)] = \sin(ny),$$
$$\Psi[\sin(ny)]|_{y=0,1} = 0.$$

Then, for any $m \in \pi\mathbb{N}$,

$$\langle \Psi[\sin(ny)], \sin(my)\rangle = \delta_{nm}(\frac{1}{k^2+(n-kt)^2} + \frac{1}{k^2+(n+kt)^2})$$
$$+ dk\left(\frac{1}{k^2+(kt+n)^2} - \frac{1}{k^2+(kt-n)^2}\right)\left(\frac{1}{k^2+(kt+m)^2} - \frac{1}{k^2+(kt-m)^2}\right)$$
$$+ i((-1)^{n+m}-1)nmkt$$
$$\cdot \frac{k^4t^4 + 2k^4t^2 + 2k^4 - 2k^2t^2(m^2+n^2) + 2k^2(m^2+n^2) + 2m^2n^2}{(k^2+(kt+m)^2)(k^2+(kt-m)^2)(k^2+(kt+n)^2)(k^2+(kt-n)^2)(n^2-m^2)},$$

where

$$d = -((-1)^{n+m}-1) + 2\frac{(-1)^{n+m}e^{-k}+e^{k}}{e^{k}-e^{-k}} + 2\frac{(-1)^{m}e^{ikt} - (-1)^{n}e^{-ikt}}{e^{k}-e^{-k}}.$$

Before proving this result, let us comment on some of the implications and the relation to the results of Section 3.

- While these coefficients are much less simple than for a Fourier basis, they asymptotically decay with rates $n^{-3}m^{-3}$. Hence, an argument via Schur's test as in (118) does not have to require $\alpha < 1$. Furthermore, the rapid decay suggests that the mapping

$$W \mapsto \Psi,$$
$$L^2 \to L^2 \tag{119}$$

  can be extended to a bounded mapping on the fractional Sobolev spaces:

$$\sum_n n^{2s}|W_n|^2,$$

  for $s > 0$ not too large.
- Using that $n, m, kt \geq 0$, one may roughly bound

$$\frac{k^2t^2}{\sqrt{k^2+(n+kt)^2}\sqrt{k^2+(m+kt)^2}} \leq 1,$$

  and thus trade the additional decay for the convenience of a uniform bound. While this is far from optimal, it reduces estimates to the ones for the Fourier basis.
- In Section 3 we use a different approach and consider boundary terms separately. That is, we decompose $\partial_y\Phi$ into a function, $\Phi^{(1)}$, with zero Dirichlet conditions

$$(k^2 - (g(\partial_y - ikt))^2)\Phi^{(1)} = \partial_y W + [(g(\partial_y - ikt))^2, \partial_y]\Phi,$$
$$\Psi^{(1)}|_{y=0,1} = 0,$$

and a *homogeneous correction*

$$
\begin{aligned}
(k^2-(\partial_y-ikt)^2)H^{(1)}&=0,\\
H^{(1)}|_{y=0,1}&=\partial_y\Phi|_{y=0,1}.
\end{aligned}
$$

The estimate of

$$\partial_y W+[(g(\partial_y-ikt))^2,\partial_y]\Phi\mapsto \Phi^{(1)}$$

is then similar to the estimate of Φ in terms of W. In order to control $H^{(1)}$, we make additional use of the dynamics and study the evolution of

$$\partial_y\Phi|_{y=0,1}.$$

- We further note, that by our choice of basis, for $\frac{1}{2}<s<1$, $\partial_y W\in H^s$ would also imply that $\partial_y W|_{y=0,1}$ vanishes for all times. However, $\partial_y W|_{y=0,1}$ is not conserved by the linearized Euler equations.

Proof of Lemma 4.15. The streamfunction $\Psi[\sin(ny)]$ is given by

$$
\begin{aligned}
\Psi[\sin(ny)]=&\left(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2}\right)\sin(ny)\\
&+i\left(\frac{1}{k^2+(n-kt)^2}-\frac{1}{k^2+(n+kt)^2}\right)\left(\cos(ny)+ae^{ky+ikty}+be^{-ky+ikty}\right),
\end{aligned}
$$

where a,b solve

$$\begin{pmatrix} e^{k+ikt} & e^{-k+ikt}\\ 1 & 1\end{pmatrix}\begin{pmatrix} a\\ b\end{pmatrix}=-\begin{pmatrix} (-1)^n\\ 1\end{pmatrix}.$$

Integrating against another basis function, $\sin(my)$, we obtain:

$$
\begin{aligned}
&\langle\Psi[\sin(ny)],\sin(my)\rangle\\
&=\delta_{nm}\left(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2}\right)+i\left(\frac{1}{k^2+(n-kt)^2}-\frac{1}{k^2+(n+kt)^2}\right)\\
&\cdot\left(\frac{m((-1)^{n+m}-1)}{n^2-m^2}+\frac{1}{2i}\left(\frac{1}{k+i(kt+m)}-\frac{1}{k+i(kt-m)}\right)((-1)^m e^{k+ikt}-1)a\right.\\
&\left.+\frac{1}{2i}\left(\frac{1}{-k+i(kt+m)}-\frac{1}{-k+i(kt-m)}\right)((-1)^m e^{-k+ikt}-1)b\right).
\end{aligned}
$$

As the δ_{nm} term is already of the desired form, in the following we consider only the remaining terms. Using the equation for a,b, we obtain

$$
\begin{aligned}
&\left(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2}\right)\left(\frac{im((-1)^{n+m}-1)}{n^2-m^2}\right.\\
&-\frac{1}{2}\left(\frac{1}{k+i(kt+m)}-\frac{1}{k+i(kt-m)}+\frac{1}{-k+i(kt+m)}-\frac{1}{-k+i(kt-m)}\right)((-1)^{n+m}-1)\\
&\left.+\frac{1}{2}\left(\frac{1}{k+i(kt+m)}-\frac{1}{k+i(kt-m)}-\frac{1}{-k+i(kt+m)}+\frac{1}{-k+i(kt-m)}\right)d\right),
\end{aligned}
\tag{120}
$$

where

$$
\begin{aligned}
d&=((-1)^m e^{k+ikt}-1)a-((-1)^m e^{-k+ikt}-1)b\\
&=((-1)^{n+m}-1)-2((-1)^m e^{-k+ikt}-1)b\\
&=-((-1)^{n+m}-1)+2\frac{((-1)^m e^{-k+ikt}-1)((-1)^n-e^{k+ikt})}{e^{k+ikt}-e^{-k+ikt}}\\
&=-((-1)^{n+m}-1)+2\frac{(-1)^{n+m}e^{-k}+e^k}{e^k-e^{-k}}+2\frac{(-1)^m e^{ikt}-(-1)^n e^{-ikt}}{e^k-e^{-k}}.
\end{aligned}
$$

We, in particular, note that

$$d(t,k,n,m)=\overline{d(t,k,m,n)}=d(-t,k,m,n),$$

and that d is uniformly bounded if k is bounded away from zero. Furthermore, consider k large and $n+m$ even, then in (120) only the contribution involving d is present and

$$\begin{aligned}(-1)^{n+m}-1&=0,\\ d=2\frac{e^k}{e^k-e^{-k}}+\mathcal{O}(e^{-k})&=2+\mathcal{O}(e^{-k})>1.\end{aligned}$$

The factor in front of d and $((-1)^{n+m}-1)$, in (120), are given by the real and imaginary part of

$$\begin{aligned}&\frac{1}{k+i(kt+m)}-\frac{1}{k+i(kt-m)}=\frac{k-i(kt+m)}{k^2+(kt+m)^2}-\frac{k-i(kt-m)}{k^2+(kt-m)^2}\\ &=(k-ikt)(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2})-im(\frac{1}{k^2+(kt+m)^2}+\frac{1}{k^2+(kt-m)^2}).\\ &=k(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2})+i\frac{kt(4ktm)-m(2k^2+2k^2t^2+2m^2)}{(k^2+(kt-m)^2)(k^2+(kt-m)^2)}\\ &=k(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2})-i\frac{m(2k^2-2k^2t^2+2m^2)}{(k^2+(kt-m)^2)(k^2+(kt-m)^2)}.\end{aligned}$$

The coefficients $c_{nm}(t,k)$ are hence explicitly given by:

$$\begin{aligned}&\delta_{nm}(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2})\\ &\quad+dk\left(\frac{1}{k^2+(kt+n)^2}-\frac{1}{k^2+(kt-n)^2}\right)\left(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2}\right)\\ &\quad+i((-1)^{n+m}-1)\\ &\quad\cdot\left(\frac{1}{k^2+(kt+n)^2}-\frac{1}{k^2+(kt-n)^2}\right)\left(-\frac{m(2k^2-2k^2t^2+2m^2)}{(k^2+(kt+m)^2)(k^2+(kt-m)^2)}+\frac{m}{n^2-m^2}\right)\\ &=\delta_{nm}(\frac{1}{k^2+(n-kt)^2}+\frac{1}{k^2+(n+kt)^2})\\ &\quad+dk\left(\frac{1}{k^2+(kt+n)^2}-\frac{1}{k^2+(kt-n)^2}\right)\left(\frac{1}{k^2+(kt+m)^2}-\frac{1}{k^2+(kt-m)^2}\right)\\ &\quad+i((-1)^{n+m}-1)\\ &\quad\cdot nmkt\frac{k^4t^4+2k^4t^2+2k^4-2k^2t^2(m^2+n^2)+2k^2(m^2+n^2)+2m^2n^2}{(k^2+(kt+m)^2)(k^2+(kt-m)^2)(k^2+(kt+n)^2)(k^2+(kt-n)^2)(n^2-m^2)}.\end{aligned}$$

□

6. Stability and boundary conditions

In Section 3.3, we required ω_0 to satisfy zero Dirichlet conditions to establish decay of $\partial_y^2\Phi$ and $\partial_y^2\Psi$. In this section, we show that this conditions is necessary, both for the explicit example

$$\omega_0(x,y)=2i\sin(x),\quad (x,y)\in\mathbb{T}_\pi\times[0,1],$$

as well as for general functions with $\hat{\omega}_0(k)\in H^2([0,1])$. For simplicity, we here only consider linearized Couette flow. Results for general monotone shear flows are obtained in Chapter 5.

LEMMA 4.16. *Consider the linearized Couette flow in scattering formulation*

$$\partial_t W = 0,$$
$$(-k^2 + (\partial_y - ikt)^2)\Psi = W,$$

with initial datum $\hat{\omega}_0(k,y) = \delta_1(k) - \delta_{-1}(k)$. Then there exists a sequence $t_n \to \infty$, such that $e^{-kt_n y}\partial_y^2 \Psi(t_n,k,y)$ converges to a non-trivial limit in L^2_y.

PROOF. By symmetry it suffices to consider $k = 1$. The stream function Ψ is then given by

$$\frac{1}{1+t^2}(-1 + a(t)e^{y+ity} + b(t)e^{-y+ity}),$$

where a, b solve

$$\begin{pmatrix} e^{1+it} & e^{-1+it} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Differentiating twice, we obtain

$$\Xi := e^{-ity}\partial_y^2 \Psi = \frac{(1+it)^2}{1+t^2} a(t)e^y + \frac{(-1+it)^2}{1+t^2} b(t)e^{-y}.$$

As $a(t), b(t)$ depend on t only via e^{it}, for any $c \in \mathbb{R}, m \in 2\pi\mathbb{Z}$

$$a(c) = a(c+m),$$
$$b(c) = b(c+m).$$

We may thus, for example, consider sequences $t_{1,n} \in 2\pi\mathbb{Z}$ and $t_{2,n} \in 2\pi\mathbb{Z}+\pi$ tending to $\pm\infty$. Along these sequences a, b are constant and non-trivial, while

$$\frac{(\pm 1 + it)^2}{1+t^2} \to -1.$$

Therefore,

$$\Xi(t_n) \to -ae^y - be^{-y} \neq 0,$$

which yields the desired result.

□

A similar result also holds for generic ω_0:

LEMMA 4.17. *Consider the linearized Couette flow in scattering formulation*

$$\partial_t W = 0,$$
$$(-k^2 + (\partial_y - ikt)^2)\Psi = W.$$

Let further $\hat{\omega}_0(k,\cdot) \in H^2([0,1])$ and suppose that for some $k \neq 0$, $\hat{\omega}_0(k,\cdot)|_{y=0,1}$ is non-trivial. Then $e^{-ity}\partial_y^2\Psi$ does not converge to zero in L^2 as $t \to \pm\infty$.

PROOF. Splitting $\partial_y^2 \Psi = \Psi^{(2)} + H^{(2)}$ as in Section 3.3, we obtain

$$\|\Psi^{(2)}\|_{L^2}^2 \leq \|\Psi^{(2)}\|_{\tilde{H}^1}^2 = \langle \Psi^{(2)}, \partial_y^2 W \rangle \leq \sum_n < \frac{n}{k} - t >^{-2} |(\partial_y^2 W)_n|^2.$$

Using a similar argument as in the proof of Theorem 4.4, one can show that $\|\Psi^{(2)}\|_{L^2} \to 0$. Here we use that, for Couette flow, W is preserved in time and hence an L^2 estimate suffices. In the more general case, for this argument one would either need some additional control of the L^2 integrability, e.g.

$$\lim_{N\to\infty} \sup_{t>0} \sum_{|n|\geq N} |(\partial_y^2 W)_n|^2 = 0,$$

or control in a fractional Sobolev space. This is further discussed in Chapter 5.

It thus remains to consider

$$e^{-ikty}H^{(2)} = \partial_y^2\Psi(0)e^{-ikty}u_1 + \partial_y^2\Psi(1)e^{-ikty}u_2.$$

For convenience of notation, we again set $k=1$.

Restricting sequences $t_n \in 2\pi\mathbb{N}$, $e^{-ity}u_1$ and $e^{-ity}u_2$ do not depend on t and are linearly independent. It thus suffices to show that $\partial_y^2\Psi(0)$ and $\partial_y^2\Psi(1)$ cannot both converge to zero unless ω_0 satisfies zero Dirichlet conditions.

Solving

$$(-1+(\partial_y - it)^2)\Psi = \hat{\omega}_0,$$

for $\partial_y^2\Psi$, we obtain

$$\partial_y^2\Psi|_{y=0,1} = \hat{\omega}_0|_{y=0,1} + 2it\partial_y\Psi|_{y=0,1}.$$

Testing the above equation with u_j, yields

$$\begin{aligned}\partial_y\Psi|_{y=0,1} &= \langle \hat{\omega}_0, u_j\rangle = \langle \hat{\omega}_0, e^{ity}(ae^y + be^{-y})\rangle \\ &= \frac{1}{it}\hat{\omega}_0|_{y=0,1} - \frac{1}{it}\int e^{ity}\partial_y\left(\hat{\omega}_0(ae^y+be^{-y})\right)\rangle \\ &= \frac{1}{it}\hat{\omega}_0|_{y=0,1} + \|\hat{\omega}_0\|_{H^2}\mathcal{O}(t^{-2}).\end{aligned}$$

Here we used that $e^{it_ny}|_{y=0,1} = 1$ for our sequence of t_n. Therefore,

$$\partial_y^2\Psi|_{y=0,1} = 3\hat{\omega}_0|_{y=0,1} + \mathcal{O}(t_n^{-1}) \not\to 0,$$

which concludes the proof. $\square$

Using the same approach, one can obtain similar results for higher Sobolev norms involving boundary values of higher derivatives. Here one can consider both $\omega_0 = \sin(x)P(y)$, P a polynomial, and general ω_0. However, for non-Couette flow the boundary values of higher derivatives are not conserved by the evolution and therefore conditions of the form

$$\partial_y^n W|_{y=0,1} \equiv 0$$

are in general never satisfied for $n \geq 1$. Instead, one would have to derive necessary and sufficient conditions under which $\partial_y^n W|_{y=0,1} \to 0$ as $t \to \pm\infty$.

In the following Chapter 5 we prove that such restrictions are also necessary for general monotone shear flows that the critical fractional Sobolev space is given by $H_y^{\frac{3}{2}}$. That is, we prove stability in all subcritical periodic fractional Sobolev spaces $H_y^s(\mathbb{T}), s < \frac{3}{2}$, and blow-up in all supercritical Sobolev space and thus, in particular, that H^2 stability can not hold in general, unless one restricts to perturbations with zero Dirichlet data, $\omega_0|_{y=0,1} = 0$, where the critical space is given by $H_y^{\frac{5}{2}}$.

CHAPTER 5

Boundary effects and sharp stability results

In the previous Chapter 4, we proven that the linearized 2D Euler equations in a finite periodic channel, $\mathbb{T}_L \times [0,1]$, are stable in $H^m_x H^1_y(\mathbb{T}_L \times [0,1])$ for general perturbations, but only stable in $H^m_x H^2_y(\mathbb{T}_L \times [0,1])$ under perturbations with zero Dirichlet boundary data, $\omega_0|_{y=0,1} = 0$.

In this Chapter, we study the boundary effects and the associated singularity formation in more detail and show that the critical Sobolev spaces in y are given by $H^{\frac{3}{2}}_y$ and $H^{\frac{5}{2}}_y$, respectively. More precisely, we show that stability in $H^m_x H^s_y(\mathbb{T}_L \times [0,1]), s > \frac{3}{2}$ can not hold for general perturbations due the development of (logarithmic) singularities at the boundary. On the other hand, we prove stability in $H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T})$ for any $s < \frac{3}{2}$, where for technical reasons we consider periodic fractional Sobolev spaces, $H^s(\mathbb{T})$, instead of $H^s([0,1])$. In particular, stability in $H^s, s > 1$, allows us to prove damping with an integrable rate and thus extend the scattering result of the previous chapter to initial perturbations without zero Dirichlet data, which has not been possible with the H^1 stability results of Chapter 4.

Restricting to perturbations with zero Dirichlet boundary data, $\omega_0|_{y=0,1} = 0$, we similarly show that the critical space is given by $H^{\frac{5}{2}}$ and prove stability and instability for $H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T}), s < \frac{5}{2}$, and $H^m_x H^s_y(\mathbb{T}_L \times [0,1]), s > \frac{5}{2}$, respectively. As we discuss in Chapter 6, these improvements allow us to study consistency of the nonlinear problem in the finite periodic channel, where the singularity formation at the boundary and the resulting regularity restrictions have a large effect on possible nonlinear damping results.

1. Fractional Sobolev spaces

As we make extensive use of fractional Sobolev spaces, we provide a short introduction to their various definitions and properties. Here we follow [**DNPV11**] (published as [**DNPV12**]).

In the whole space, fractional Sobolev spaces can be equivalently characterized using either a Fourier weight or an appropriate kernel:

PROPOSITION 5.1 (Fractional Sobolev space on $\mathbb{R}$; [**DNPV11**, Section 3]). *Let $0 < s < 1$, then there exists C_s such that for any $u \in \mathcal{S}(\mathbb{R})$*

$$\| |\eta|^s \mathcal{F} u\|^2_{L^2} = C_s \iint_{\mathbb{R}\times\mathbb{R}} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}} dxdy.$$

In particular, both expressions define the same quasi-norm. The fractional Sobolev space, $H^s(\mathbb{R})$, is then defined as the closure of $\mathcal{S}(\mathbb{R})$ with respect to

$$\|u\|^2_{L^2} + \| |\eta|^s \mathcal{F} u\|^2_{L^2}.$$

$H^s(\mathbb{R})$ is a Hilbert space with inner product

$$\begin{aligned}\langle u, v\rangle_{H^s} &= \langle u, v\rangle_{L^2} + \langle |\eta|^{s/2}\mathcal{F}u, |\eta|^{s/2}\mathcal{F}v\rangle_{L^2} \\ &= \langle u, v\rangle_{L^2} + C_s \iint_{\mathbb{R}\times\mathbb{R}} \frac{(u(x)-u(y))\overline{(v(x)-v(y))}}{|x-y|^{1+2s}} dxdy.\end{aligned}$$

For $s > 1, s \notin \mathbb{N}$, the fractional Sobolev space is (recursively) defined by requiring that $u \in H^{s-1}$ and $\partial_x u \in H^{s-1}$. The definition via a kernel can be adapted to other and higher dimensional domains. We, in particular, are interested in the setting of the interval $[0,1]$.

Proposition 5.2 (Trace map; [**DNPV11**, Section 3]). *Let $0 < s < 1$ and define $H^s([0,1])$ as the closure of $C^\infty([0,1])$ with respect to*

$$\iint_{[0,1]^2} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}} dxdy + \|u\|^2_{L^2([0,1])}.$$

Then $H^s([0,1])$ is a Hilbert space. Let further $s > 1/2$, then H^s embeds into C^0, in particular there exists a trace map and

$$|u_{y=0,1}| \lesssim_s \|u\|_{H^s([0,1])}.$$

A closely related space is given by the periodic fractional Sobolev space $H^s(\mathbb{T})$.

Proposition 5.3 ([**BO13**]). *Let $0 < s < 1/2$, then for any $u \in C^\infty(\mathbb{T})$,*

$$\||\eta|^s \mathcal{F}u\|^2_{L^2} \lesssim \iint_{\mathbb{T}\times[-\frac{1}{2},\frac{1}{2}]} \frac{|u(x+y)-u(y)|^2}{|x|^{1+2s}} dxdy \lesssim \||\eta|^s \mathcal{F}u\|^2_{L^2}.$$

In particular, both the kernel and Fourier characterization define the same quasi-norm. Furthermore,

$$\iint_{\mathbb{T}\times[-\frac{1}{2},\frac{1}{2}]} \frac{|u(x+y)-u(y)|^2}{|x|^{1+2s}} dxdy = \langle \mathcal{F}u, B_n |n|^{2s} \mathcal{F}u\rangle_{l^2},$$

where B_n satisfies

$$1 \lesssim B_n := |n|^{-2s} \int_{[-\frac{1}{2},\frac{1}{2}]} \frac{\sin^2(xn)}{4|x|^{1+2s}} dx \lesssim 1.$$

The fractional Sobolev space $H^s(\mathbb{T})$ is defined as the closure of $C^\infty(\mathbb{T})$ with respect to

$$\|u\|^2_{L^2} + \iint_{\mathbb{T}\times[-\frac{1}{2},\frac{1}{2}]} \frac{|u(x+y)-u(y)|^2}{|x|^{1+2s}} dxdy.$$

$H^s(\mathbb{T})$ is a Hilbert space, where the inner product can be chosen as either

$$\begin{aligned}\langle u, v\rangle_{H^s(\mathbb{T})} &:= \langle u, v\rangle_{L^2} + \langle \mathcal{F}u, B_n |n|^{2s} \mathcal{F}v\rangle_{l^2} \\ &= \langle u, v\rangle_{L^2} + \iint \frac{\overline{(u(x+y)-u(y))}(v(x+y)-v(y))}{|x|^{1+2s}} dxdy,\end{aligned}$$

or

$$\langle u, v\rangle_{H^s(\mathbb{T})} := \langle u, v\rangle_{L^2} + \langle \mathcal{F}u, |n|^{2s} \mathcal{F}v\rangle_{l^2}.$$

From the kernel characterization, it can easily be seen that $H^s(\mathbb{T}) \subset H^s([0,1])$:

Proposition 5.4. *Let $0 < s < 1$, then any $u \in H^s(\mathbb{T})$ is also in $H^s([0,1])$ and*

$$\|u\|_{H^s([0,1])} \lesssim \|u\|_{H^s(\mathbb{T})}.$$

Proof of Proposition 5.4. The L^2 norms are equal, hence we only have to consider the quasi-norm in $H^s([0,1])$. Introducing a change of variables $x \mapsto z + y$, we compute

$$\begin{aligned}\|u\|^2_{\dot{H}^s([0,1])} &= \iint_{[0,1]^2} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}} dxdy \\ &= \int_{[0,1]} \int_{[0,1]-y} \frac{|u(z+y)-u(y)|^2}{|z|^{1+2s}} dzdy \\ &\leq \int_{[0,1]} \int_{[-1,2]} \frac{|u(z+y)-u(y)|^2}{|z|^{1+2s}} dzdy \\ &\leq \|u\|^2_{\dot{H}^s(\mathbb{T})} + C\|u\|^2_{L^2} \lesssim \|u\|^2_{H^s(\mathbb{T})},\end{aligned}$$

where we used that

$$\sup_{|z|\geq \frac{1}{2}} \frac{1}{|z|^{1+2s}} \leq 2.$$

□

As a simplification, for the stability results of Section 2 and Section 3, we restrict to fractional Sobolev spaces, $H^s(\mathbb{T})$, in order to be able to use the Fourier characterization. In this case, we further require that the coefficient functions, f, g, corresponding to the shear flow, U, are not only sufficiently regular, e.g. $g \in W^{1,\infty}([0,1])$, but can be periodically extended in a regular way, e.g. $g \in W^{1,\infty}(\mathbb{T})$, in order to be able to apply the following Propositions 5.5 and 5.6.

Proposition 5.5 (Multiplication with Lipschitz functions). *Let $g \in W^{1,\infty}(\mathbb{T})$ be periodic and Lipschitz, then for any $s < 1/2$ and any $u \in H^s(\mathbb{T})$, also $gu \in H^s(\mathbb{T})$ and*

$$\|ug\|_{H^s} \leq \|g\|_{W^{1,\infty}} \|u\|_{H^s}.$$

Proposition 5.6 (Commutator Estimate). *Let $g \in C^{0,1}(\mathbb{T})$ with $g^2 > c > 0$ and let $0 < s < 1/2$. Then for any $u \in H^s(\mathbb{T})$*

$$\Re\langle u, g^2 u\rangle_{H^s(\mathbb{T})} \geq c\|u\|^2_{H^s(\mathbb{T})} - C_s\|g^2\|_{\dot{C}^{0,1}}\|u\|^2_{L^2}.$$

Remark 9. *The periodicity assumption on g drastically simplifies calculations, but can probably be relaxed.*

It can be shown that the multiplication with the characteristic function of the positive half-line, $1_{[0,\infty)}$, is a bounded operator on $H^s(\mathbb{R}), s < \frac{1}{2}$ (see [**RS96**, page 208]*). Thus, one can probably allow for a jump discontinuity of the periodic extension of g in Proposition 5.5 and only require that $g \in W^{1,\infty}([0,1])$.*

In the case of Proposition 5.6, we, however, use that the commutator

$$u \mapsto [(-\Delta)^{\frac{s}{2}}, g]u,$$

where $(-\Delta)^{\frac{s}{2}}$ is defined as the Fourier multiplier

$$u \mapsto \mathcal{F}^{-1}|\eta|^s \mathcal{F}u,$$

is not only a bounded operator from H^s to L^2, but gains regularity in the sense that it also is a bounded operator from $H^{s-\epsilon}$ to L^2 for some $\epsilon > 0$. As this is not the case for functions with jump discontinuities, the current proof can probably only be extended to functions g, for which the size of the jump discontinuity

$$|g^2(1) - g^2(0)|$$

is sufficiently small compared to $\min(g^2) > 0$*, so that the possible loss due to the jump satisfies (by the improved version of Proposition 5.5)*

$$|g^2(1) - g^2(0)| \|1_{[\frac{1}{2},1]} u\|_{H^s}^2 \leq \frac{\min(g^2)}{2} \|u\|_{H^s}^2$$

and can hence be absorbed by

$$\langle u, \min(g^2) u\rangle_{H^s} = \min(g^2) \|u\|_{H^s}^2.$$

Removing the restriction on the size of the jump,

$$|g^2(1) - g^2(0)|,$$

is probably possible, but would require considerable additional technical effort.

Proof of Proposition 5.5. We remark that $gu \in L^2$ and that $\|gu\|_{L^2} \leq \|g\|_{W^{1,\infty}} \|u\|_{L^2}$ is well-known. For the H^s seminorm we follow the standard proof via the kernel characterization (see [**DNPV11**, page 21]).

$$\begin{aligned}
&\iint_{\mathbb{T}\times[-\frac{1}{2},\frac{1}{2}]} \frac{|u(x+y)g(x+y) - u(y)g(y)|^2}{|x|^{1+2s}} dxdy \\
&\lesssim \iint_{\mathbb{T}\times[-\frac{1}{2},\frac{1}{2}]} |g(x+y)|^2 \frac{|u(x+y) - u(y)|^2}{|x|^{1+2s}} dxdy \\
&\quad + \iint_{\mathbb{T}\times[-\frac{1}{2},\frac{1}{2}]} |u(y)|^2 \frac{|g(x+y) - g(y)|^2}{|x|^{1+2s}} dxdy.
\end{aligned}$$

The first term can be easily controlled by $\|g\|_{L^\infty}^2 \|u\|_{H^s}^2$. For the second term we use that $g \in W^{1,\infty}(\mathbb{T})$ is Lipschitz and thus

$$\frac{|g(x) - g(y)|^2}{|x-y|^{1+2s}} \leq \frac{1}{|x-y|^{2s-1}} \|g\|_{W^{1,\infty}}^2.$$

Then,

$$\sup_{y\in\mathbb{T}} \int_{[-\frac{1}{2},\frac{1}{2}]} \frac{1}{|x-y|^{2s-1}} dx \leq \int_{-1}^{2} \frac{1}{|x|^{2s-1}} dx < \infty,$$

as $1 - 2s > -1$ for all $0 < s < 1$. The second term can thus be controlled in terms of $\|u\|_{L^2}^2 \|g\|_{W^{1,\infty}}^2$. □

Proof of Proposition 5.6. For the L^2 product there is nothing to show.
By the kernel characterization

$$\begin{aligned}
\Re\langle u, g^2 u\rangle_{H^s(\mathbb{T})} &= \Re \iint \frac{\overline{(u(x+y) - u(y))}(g^2(x+y)u(x+y) - g^2(y)u(y))}{|x|^{1+2s}} dxdy \\
&= \iint g^2(x+y) \frac{|u(x+y) - u(y)|^2}{|x|^{1+2s}} dxdy \\
&\quad - \Re \iint \frac{g^2(x+y) - g^2(y)}{|x|^{1+2s}} \overline{(u(x+y) - u(y))} u(y) dxdy.
\end{aligned}$$

As g^2 is Lipschitz, the second term can thus be estimated by

$$\begin{aligned}
&\|g^2\|_{W^{1,\infty}} \iint \frac{1}{|x|^{2s}} |u(x+y) - u(y)||u(y)| dxdy \\
&\leq 2\|g^2\|_{W^{1,\infty}} \left\| \frac{1}{|x|^{2s}} \right\|_{L_x^1} \|u\|_{L^2}^2 \leq C_s \|g^2\|_{W^{1,\infty}} \|u\|_{L^2}^2,
\end{aligned}$$

where we used that $2s < 1$. □

2. Stability in $H^{3/2-}$ and boundary perturbations

In Chapter 4 we established stability of the linearized Euler equations, (90), in a finite periodic channel, $\mathbb{T}_L \times [0,1]$, in $H^m_x H^1_y$, for general initial data. The damping result, Theorem 4.4, hence provides decay of the perturbations to the velocity field with rate t^{-1}, i.e.

$$
\begin{aligned}
\|v - \langle v \rangle_x\|_{L^2_{x,y}(\mathbb{T}_L\times[0,1])} &= \mathcal{O}(t^{-1}),\\
\|v_2\|_{L^2_{x,y}(\mathbb{T}_L\times[0,1])} &= \mathcal{O}(t^{-1}).
\end{aligned} \tag{121}
$$

As this is almost sufficient to establish scattering, a natural question to ask is how far this can be improved, that is for which values of s, with $s > 1$, stability in $H^m_x H^s_y$ still holds.

As the main result of this section, we show that the critical Sobolev exponent in y is given by $s = \frac{3}{2}$. More precisely, in the Corollaries 5.1 and 5.2, we show that for perturbations ω_0 with non-vanishing Dirichlet data, $\omega_0|_{y=0,1}$, $\partial_y W$ asymptotically develops (logarithmic) singularities at the boundary and that hence stability in $H^m_x H^s_y(\mathbb{T}_L \times [0,1]), s > \frac{3}{2}$, and $H^m_x H^2_y(\mathbb{T}_L \times [0,1])$ can not hold, unless one restricts to perturbations ω_0 such that $\omega_0|_{y=0,1} = 0$. This singularity formation is further analyzed in Section 4, where we also study the behavior close to the boundary and the heuristic implications for stability in L^p spaces. As we discuss in Chapter 6, these instability results have strong implications for the problem of nonlinear inviscid damping in a finite channel.

As a complementary result to the singularity formation, Theorem 5.1 establishes stability in the *periodic* fractional Sobolev spaces $H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T}), s < 3/2$. In particula, we thus obtain inviscid damping with an integrable (but subquadratic) rate and hence scattering for initial perturbations without zero Dirichlet data, which has not been possible with the H^1 stability results of Chapter 4.

We recall that the linearized 2D Euler equations, (91), in a finite periodic channel, $\mathbb{T}_L \times [0,1]$, are given by:

$$
\begin{aligned}
\partial_t W &= \frac{if(y)}{k}\Phi,\\
(-1 + (g(y)(\frac{\partial_y}{k} - it))^2)\Phi &= W,\\
\Phi|_{y=0,1} &= 0,\\
&(t,k,y) \in \mathbb{R} \times L(\mathbb{Z}\setminus\{0\}) \times [0,1].
\end{aligned} \tag{122}
$$

Furthermore, as noted in Remark 3 of Chapter 4, the equations (122) decouple with respect to k. Hence, for the remainder of this chapter, we consider k as a given parameter and consider the stability of

$$
W(t) = W(t,k,\cdot) \in H^s([0,1]).
$$

Results for $H^m_x H^s_y(\mathbb{T}_L \times [0,1]), m \in \mathbb{N}_0$, can then be obtained by summing over k.

Considering the evolution of $\partial_y W$:

$$
\begin{aligned}
\partial_t \partial_y W &= \frac{if}{k}\partial_y \Phi + \frac{if'}{k}\Phi, \\
(-1+(g(\frac{\partial_y}{k}-it))^2)\Phi^{(1)} &= \partial_y W + [(g(\partial_y - it))^2, \partial_y]\Phi, \\
\Phi^{(1)}_{y=0,\pi} &= 0, \\
H^{(1)} &= \partial_y \Phi - \Phi^{(1)}, \\
(t,k,y) &\in \mathbb{R} \times L(\mathbb{Z}\setminus\{0\}) \times [0,1],
\end{aligned} \tag{123}
$$

at the boundary, $y \in \{0,1\}$, we prove that Sobolev stability can not hold for $s > \frac{3}{2}$, unless one restricts to perturbations ω_0 with $\omega_0|_{y=0,1} \equiv 0$. In that case, as we show in Section 3, an instability develops for $s > \frac{5}{2}$.

The following lemma recalls the explicit characterization of $\partial_y \Phi|_{y=0,1}$, which has been obtained in the proof of Lemma 4.7, and describes the asymptotic behavior of $\partial_y \Phi|_{y=0,1}$.

LEMMA 5.1. *Let W be a solution of the linearized Euler equations,* (122)*, and suppose that $g \in W^{2,\infty}([0,1])$ satisfies $g^2 > c > 0$. Then,*

$$
\begin{aligned}
\partial_y \Phi|_{y=0} &= \frac{k}{g^2(0)}\langle W, u_1 \rangle, \\
\partial_y \Phi|_{y=1} &= \frac{k}{g^2(1)}\langle W, u_2 \rangle,
\end{aligned} \tag{124}
$$

where

$$
\begin{aligned}
u_1(t,y) &= e^{ikty}u_1(0,y), \\
u_2(t,y) &= e^{ikt(y-1)}u_2(0,y),
\end{aligned}
$$

and $u_j(0,y)$ are solutions of

$$
\begin{aligned}
(-k^2 + (g\partial_y)^2)u &= 0, \\
y &\in [0,1],
\end{aligned}
$$

with boundary values

$$
\begin{aligned}
u_1(0,0) = u_2(0,1) &= 0, \\
u_2(0,1) = u_2(0,0) &= 0.
\end{aligned} \tag{125}
$$

Let $s > 0$ and suppose that

$$
\|\partial_y W(t)\|_{H^s} < C < \infty
$$

for all time, then, as $t \to \infty$,

$$
\langle W, u_j \rangle|_{j=1,2} = \frac{1}{t}\omega_0|_{y=0,1} + \mathcal{O}(t^{-1-s}).
$$

As a corollary, we see that stability in $s > 3/2$ can in general not hold.

COROLLARY 5.1. *Let W be a solution of the linearized Euler equations,* (122)*, and suppose that $f, g \in W^{2,\infty}([0,1])$ and that g satisfies $g^2 > c > 0$. Let $s > 1$ and suppose that*

$$
\|\partial_y W\|_{H^s([0,1])} < C < \infty.
$$

Suppose further that $f\omega_0|_{y=0,1}$ is non-trivial. Then

$$
\|\partial_y W(t)\|_{L^\infty([0,1])} \gtrsim \log|t|
$$

as $t \to \pm\infty$.

As a consequence, for perturbations such that $f\omega_0|_{y=0,1}$ is non-trivial, for any $s > \frac{3}{2}$, necessarily

$$\sup_{t>0} \|W(t)\|_{H^s([0,1])} = \infty.$$

Proof of Corollary 5.1. Restricting (123) to the boundary, we obtain

$$\partial_t \partial_y W|_{y=0,1} = \frac{if}{k} \partial_y \Phi|_{y=0,1},$$

where we used that $\Phi|_{y=0,1} = 0$.

By Lemma 5.1, under the assumptions of the corollary, thus

$$\partial_t \partial_y W|_{y=0,1} = \frac{1}{t} \frac{if}{k} \omega_0 \frac{k}{g^2}\bigg|_{y=0,1} + \mathcal{O}(t^{-1-s}).$$

Integrating this equality and using that

$$\frac{if}{k} \omega_0 \frac{k}{g^2}\bigg|_{y=0,1}$$

is independent of t and non-trivial,

$$|\partial_y W|_{y=0,1}(t)| \gtrsim \int^t \frac{1}{\tau} - \mathcal{O}(\tau^{-1-s}) d\tau \gtrsim \log|t|,$$

which provides the lower bound on $\|\partial_y W\|_{L^\infty}$ and hence the first result.

The second result is proven by contradiction. Let thus $s > 3/2$ be given and suppose to the contrary that

$$\|W(t)\|_{H^s} < C < \infty,$$

uniformly in time. Then, by the trace map and the first result,

$$\log(t) \lesssim \|\partial_y W\|_{L^\infty} \lesssim_s \|W(t)\|_{H^s} < C,$$

which is a contradiction as $t \to \infty$. □

Proof of Lemma 5.1. The explicit characterization, (124), has been obtained in the proof of Lemma 4.7.

Integrating

$$u_1(t,y) = e^{ikty} u_1(0,y) = u_1(0,1) \partial_y \frac{e^{ikty}}{ikt}$$

by parts, we obtain a boundary term

$$\frac{1}{ikt} W u_1|_{y=0,1} = -\frac{1}{ikt} W|_{y=0} = -\frac{1}{ikt} \omega_0|_{y=0},$$

as well as a bulk term

$$\frac{1}{ikt} \langle e^{ikty}, \partial_y (W u_1(0,y)) \rangle = \frac{1}{ikt} \langle e^{ikty} u_1, \partial_y W \rangle + \frac{1}{ikt} \langle e^{ikty} \partial_y u_1, W \rangle.$$

The boundary term is already of the desired form.

The second term of the bulk contribution can be integrated by parts once more and thus yields a quadratically decaying contribution. It thus remains to estimate the first term,

$$\frac{1}{ikt} \langle e^{ikty} u_1, \partial_y W \rangle.$$

There we use duality and estimate

$$\langle e^{ikty} u_1, \partial_y W \rangle_{L^2} \leq \|e^{ikty} u_1\|_{H^{-s}} \|\partial_y W\|_{H^s} = \mathcal{O}(t^{-s}) \|\partial_y W\|_{H^s}.$$

□

As a consequence we obtain an improvement of the results of Section 6 of Chapter 4 and show that for stability in H^2 it is necessary to restrict to perturbations with vanishing Dirichlet boundary data, $\omega_0|_{y=0,1} = 0$.

COROLLARY 5.2. *Let $\omega_0 \in H^2, f, g$ satisfy the assumptions of Lemma 5.1 and suppose that $f\omega_0|_{y=0,1}$ is non-trivial. Let further $W(t)$ be the solution of the linearized Euler equations,* (122). *Then,*

$$\sup_t \|W(t)\|_{H^2([0,1])} = \infty.$$

PROOF OF COROLLARY 5.2. We follow the same strategy as in the proof of Corollary 5.1. Thus, assume to the contrary that $\|W(t)\|_{H^2}$ is bounded uniformly in time. Then, for example at $y = 0$,

$$\begin{aligned}\langle W, e^{ity}u_1\rangle_{L^2} &= \frac{1}{ikt}W|_{y=0} - \frac{1}{ikt}\langle e^{ikty}, \partial_y(Wu_1)\rangle_{L^2}, \\ &= \frac{1}{ikt}W|_{y=0} + \frac{1}{k^2t^2}\partial_y(Wu_1)|_{y=0}^1 - \frac{1}{k^2t^2}\langle e^{ikty}, \partial_y^2(Wu_1)\rangle_{L^2}.\end{aligned}$$

Both the last L^2 product and the trace of W and $\partial_y W$ can be controlled by $\|W\|_{H^2([0,1])}$. Thus,

$$\partial_y\Phi|_{y=0} = \frac{k}{g^2(0)}\langle W, e^{ity}u_1\rangle_{L^2} = \frac{g^2(0)}{it}\omega_0|_{y=0} + \mathcal{O}(t^{-2})\|W\|_{H^2([0,1])},$$

where we used Lemma 5.1.

Integrating

$$\partial_t\partial_y W|_{y=0,1} = \frac{if}{k}\partial_y\Phi|_{y=0,1},$$

in t, thus yields that $\partial_y W|_{y=0,1}$ blows up logarithmically as $t \to \infty$. On the other hand, the L^∞ norm of $\partial_y W$ is controlled by the H^2 norm via the Sobolev embedding theorem, which yields a contradiction. □

We have thus seen that, in general, for the purposes of stability results s can not be larger than $3/2$. The main result of this section is that this condition is sharp in the sense that stability in H^s holds for all $s < 3/2$. More precisely, instead of $H^s([0,1])$, we consider *periodic* spaces, i.e.

$$W(t,k,\cdot) \in H^{s-1}(\mathbb{T}), \partial_y W(t,k,\cdot) \in H^{s-1}(\mathbb{T}),$$

where $\mathbb{T} = [0,1]/\sim$ is the torus of unit period. As discussed in Section 1, this allows us to use both a Fourier characterization and a kernel characterization.

THEOREM 5.1. *Let $0 < s < 1/2$, $\omega_0 \in H^1([0,1])$ and $\omega_0, \partial_y\omega_0 \in H^s(\mathbb{T})$. Suppose further that $f, g \in W^{2,\infty}(\mathbb{T})$ satisfy the assumptions of the H^2 stability result, Theorem 4.15, and that*

$$\|f\|_{W^{2,\infty}(\mathbb{T})}L$$

is sufficiently small. Then the solution, W, of the linearized Euler equations, (122), *satisfies*

$$\|\partial_y W(t)\|_{H^s(\mathbb{T})} \lesssim \|\omega_0\|_{H^s(\mathbb{T})} + \|\partial_y\omega_0\|_{H^s(\mathbb{T})},$$

uniformly in time.

REMARK 10. *The assumptions on f and g are chosen such that we can apply Proposition 5.5 to the functions f, g and their derivatives f' and g'. Furthermore, we require*

$$g^2 = U'(U^{-1}(\cdot))^2$$

to be such that we can apply Proposition 5.6.

As discussed in Remark 9, these assumptions can probably be relaxed to requiring that

$$f, g \in W^{3,\infty}([0,1]),$$

and that

$$|g^2(1) - g^2(0)| = |(U'(b))^2 - (U'(a))^2|$$

is sufficiently small compared to

$$\min(g^2) = \min((U')^2) > 0.$$

PROOF OF THEOREM 5.1. In our proof, we follow a similar approach as in Chapter 4. We split $\partial_y \Phi$ into a solution with zero Dirichlet boundary conditions and a correction term in the form of a homogeneous solution:

$$\begin{aligned}
\partial_t \partial_y W &= ikf\Phi^{(1)} + ikf'\Phi + ikfH^{(1)}, \\
(-k^2 + (g(\partial_y - ikt))^2)\Phi^{(1)} &= \partial_y W + [(g(\partial_y - ikt))^2, \partial_y]\Phi, \\
\Phi^{(1)}|_{y=0,1} &= 0,
\end{aligned}$$

where $H^{(1)}$ is given by

$$\begin{aligned}
&(-k^2 + (g(\partial_y - ikt))^2)H^{(1)} = 0, \\
&H^{(1)} = H^{(1)}|_{y=0} e^{ikty} u_1 + H^{(1)}|_{y=1} e^{ikt(y-1)} u_2, \\
&H^{(1)}|_{y=0} = \partial_y \Phi|_{y=0} = \frac{1}{g^2} \langle W, e^{ikty} u_1 \rangle, \\
&H^{(1)}|_{y=1} = \partial_y \Phi|_{y=1} = \frac{1}{g^2} \langle W, e^{ikt(y-1)} u_2 \rangle.
\end{aligned}$$

Considering a decreasing weight A and computing

$$\partial_t(\langle W, AW \rangle_{H^s} + \langle \partial_y W, A\partial_y W \rangle_{H^s}) =: \partial_t I(t),$$

we thus have to control

$$\begin{aligned}
&\text{(elliptic)} \qquad \langle \frac{if}{k}\Phi, AW \rangle_{H^s} + \langle \frac{if}{k}\Phi^{(1)}, A\partial_y W \rangle_{H^s} + \langle \frac{if'}{k}\Phi, A\partial_y W \rangle_{H^s} \\
&\text{(boundary)} \qquad + \langle \frac{if}{k} H^{(1)}, A\partial_y W \rangle_{H^s}
\end{aligned}$$

in terms of

$$\frac{C}{k} |\langle W, \dot{A}W \rangle_{H^s} + \langle \partial_y W, \dot{A}\partial_y W \rangle_{H^s}|$$

Assuming this control and requiring k to be sufficiently large such that $\frac{C}{k} \ll 1$, this then yields that $I(t)$ is non-increasing. In particular,

$$\|W\|_{H^s}^2 + \|\partial_y W\|_{H^s}^2 \lesssim I(t) \leq I(0) \lesssim \|\omega_0\|_{H^s}^2 + \|\partial_y \omega_0\|_{H^s}^2.$$

It remains to prove the elliptic and boundary control in the following subsections. □

2.1. Boundary corrections. The control of the boundary term in the proof of Theorem 5.1 is provided by the following theorem.

THEOREM 5.2. *Let $0 < s < 1/2$ and let W, f, g as in Theorem 5.1. Let further A be a diagonal operator comparable to the identity, i.e.*

$$A : e^{iny} \mapsto A_n e^{iny},$$

with

$$1 \lesssim A_n \lesssim 1,$$

uniformly in n.

Then,

$$|\langle A\partial_y W, ifH^{(1)}\rangle_{H^s}| \lesssim \sum_n c_n(t) < n >^{2s} |(\partial_y W)_n|^2,$$

for a family $c_n \in L^1_t$, with $\|c_n\|_{L^1_t}$ bounded uniformly in n.

PROOF OF THEOREM 5.2. $H^{(1)}$ is explicitly given by

$$H^{(1)} = \partial_y \Phi|_{y=0} e^{ikty} u_1 + \partial_y \Phi|_{y=1} e^{ikty(y-1)} u_2.$$

We hence have to estimate

$$\langle A\partial_y W, ifH^{(1)}\rangle_{H^s} = \partial_y \Phi|_{y=0} \langle A\partial_y W, ifu_1\rangle_{H^s} + \partial_y \Phi|_{y=1} \langle A\partial_y W, ifu_2\rangle_{H^s}. \tag{126}$$

By Lemma 5.1

$$\begin{aligned}
\partial_y \Phi|_{y=0} &= \frac{k}{g^2(0)} \langle W, e^{ikty} u_1\rangle \\
&= \frac{k}{g^2(0)} \left(\frac{1}{ikt} \omega_0|_{y=0} + \frac{1}{ikt} \langle e^{ikty}, \partial_y W u_1\rangle \right), \\
\partial_y \Phi|_{y=1} &= \frac{k}{g^2(1)} \langle W, e^{ikt(y-1)} u_2\rangle \\
&= \frac{k}{g^2(1)} \left(\frac{1}{ikt} \omega_0|_{y=1} + \frac{1}{ikt} \langle e^{ikt(y-1)}, \partial_y W u_2\rangle \right).
\end{aligned}$$

Let us for the moment concentrate on the terms not involving ω_0. Using the control of g and $\frac{1}{g}$, in order to estimate (126), we hence have to estimate

$$|\langle A\partial_y W, \frac{if}{k} e^{ikty} u_1\rangle_{H^s} \frac{1}{t} \langle \partial_y W, e^{ikty} u_1\rangle_{L^2}| \tag{127}$$

Expanding this in a basis, using that $1 \lesssim A_n \lesssim 1$, $f \in W^{1,\infty}$ and denoting

$$b_n := |(\partial_y W)_n|,$$

it suffices to consider

$$\frac{1}{t} \left(\sum_n b_n \frac{< n >^{2s}}{< n - kt >} \right) \left(\sum_n \frac{b_n}{< n - kt >} \right). \tag{128}$$

Considering the decay of the coefficients in n and taking into account that we only control $b_n < n >^s \in l^2$, we need that

$$\frac{< n >^s}{< n - kt >} \in l^2,$$

which is the case iff $s < 1/2$.

As $s < 1/2$, we may choose $\lambda < 1$ such that $s - \lambda < -1/2$ and split

$$\begin{aligned}
&\sum_n b_n \frac{< n >^s}{< n - kt >^{1-\lambda}} \frac{< n >^s}{< n - kt >^\lambda} \\
&\leq \left(\sum b_n^2 \frac{< n >^{2s}}{< n - kt >^{2(1-\lambda)}} \right)^{1/2} \left\| \frac{< n >^s}{< n - kt >^\lambda} \right\|_{l^2}.
\end{aligned}$$

Spliting the second factor in (128) in the same way, it suffices to show that

$$c_n(t) := \frac{1}{t} \frac{1}{< n - kt >^{2(1-\lambda)}} \left\| \frac{< m >^s}{< m - kt >^\lambda} \right\|_{l^2_m} \left\| \frac{1}{< m >^s < m - kt >^\lambda} \right\|_{l^2_m}$$

is in L^1_t with $\|c_n\|_{L^1_t}$ bounded uniformly in n. Estimating $< n >^s \lesssim < n - kt >^s + < kt >^s$, it suffices to show that

$$< kt >^s \left\| \frac{1}{< n >^s < n - kt >^\lambda} \right\|_{l^2} \lesssim 1.$$

As $s - \lambda < -1/2$, there exists a $\delta > 0$ such that $\lambda = 1/2 + \delta + s$. We thus estimate

$$\left\| \frac{1}{< n >^s < n - kt >^\lambda} \right\|_{l^2} \leq \| \frac{1}{< n >^s < n - kt >^s} \|_{l^\infty} \| \frac{1}{< n - kt >^{1/2+\delta}} \|_{l^2}.$$

Hence,

$$c_n(t) \lesssim \frac{1}{t} \frac{1}{< n - kt >^{2(1-\lambda)}} \in L^1_t.$$

It remains to discuss

$$\frac{1}{ikt} \omega_0|_{y=0,1}.$$

As the trace of ω_0 is controlled by its initial H^1 norm, we consider $\omega_0|_{y=0,1}$ as constants of size 1 in the following. Hence, we have to estimate

$$\left| \langle A\partial_y W, \frac{if}{k} e^{ity} u_1 \rangle \frac{1}{kt} \right|.$$

Splitting

$$\left| \frac{1}{kt} \right| = \left| \frac{1}{kt} \right|^\gamma \left| \frac{1}{kt} \right|^{1-\gamma},$$

with $1/2 < \gamma < 1/2 + \epsilon$ and using Young's inequality, we thus obtain

$$|\langle A\partial_y W, \frac{if}{k} e^{ity} u_1 \rangle \frac{1}{kt}| \lesssim < kt >^{-2\gamma} + \left| \frac{1}{kt} \right|^{2(1-\gamma)} |\langle A\partial_y W, \frac{if}{k} e^{ity} u_1 \rangle|^2.$$

Here, the first term is an integrable contribution. Following the same strategy as above, the second term can be controlled by

$$\sum_n b_n^2 \frac{< n >^{2s}}{< n - kt >^{2(1-\lambda)}} \frac{< kt >^{2s}}{< kt >^{2(1-\gamma)}}$$

Choosing γ, λ such that

$$s - (1 - \lambda) - (1 - \gamma) < -1/2$$

and modifying $c_n(t)$ to also include

$$\frac{< kt >^{2s}}{< kt >^{2(1-\gamma)} < n - kt >^{2(1-\lambda)}} \in L^1_t,$$

then proves the result. Such a choice is possible as $s < 1/2$ is given and we can choose $(1 - \lambda) < 1/2$ and $(1 - \gamma) < 1/2$ arbitrarily close to $1/2$. □

2.2. Elliptic control. In this section, our main goal is to prove the following theorem, which controls the elliptic contributions in the proof of Theorem 5.1. Here, the main steps of the proof of Theorem 5.3 are formulated as lemmata and propositions and conclude with Lemma 5.7.

Theorem 5.3. *Let $0 < s < 1/2$ and let A, f, g, W as in Theorem 5.2. Then*

$$|\langle A\partial_y W, if\Phi^{(1)} + if'\Phi \rangle_{H^s}| \lesssim \sum_n c_n(t) < n >^{2s} (|(\partial_y W)_n|^2 + |W_n|^2),$$

for a family $c_n \in L^1_t$, where $\|c_n\|_{L^1_t}$ is bounded uniformly in n.

When working with non-fractional Sobolev spaces, in Chapter 4, this estimate reduced to an elliptic regularity theorem of the form

$$\|\Phi\|_{\tilde{H}^1} \lesssim \|W\|_{\tilde{H}^{-1}},$$

where

$$\|\Phi\|_{\tilde{H}^1}^2 = \|\Phi\|_{L^2}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2}^2$$

and $\tilde{H}^{-1}$ was constructed by duality.

Similarly, we show that the proof of Theorem 5.3 reduces to estimating

$$\|\Phi\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{H^s}^2 + \|\Phi^{(1)}\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}^2.$$

Lemma 5.2. *Let $0 < s < 1/2$ and let A, f, g, W be as in Theorem 5.3. Then*

$$\begin{aligned}
&\langle A\partial_y W, if\Phi^{(1)} + if'\Phi\rangle_{H^s} \\
&\lesssim \left(\sum \frac{< n >^{2s} |(\partial_y W)_n|^2}{< n - kt >^2}\right)^{1/2} (\|f'\Phi\|_{H^s} + \|f''\Phi\|_{H^s} + \|f'(\partial_y - ikt)\Phi\|_{H^s} \\
&\quad + \|f\Phi^{(1)}\|_{H^s} + \|f'\Phi^{(1)}\|_{H^s} + \|f'(\partial_y - ikt)\Phi^{(1)}\|_{H^s}).
\end{aligned}$$

Proof of Lemma 5.2. Denote

$$R := if\Phi^{(1)} + if'\Phi.$$

Then,

$$\langle A\partial_y W, R\rangle_{H^s} = \sum_n a_n (\partial_y W)_n < n >^{2s} \langle e^{iny}, R\rangle.$$

Multiplying by a factor

$$1 = \frac{1 + i(n/k - t)}{1 + i(n/k - t)},$$

we estimate

$$\begin{aligned}
&\sum_n \left(A_n(\partial_y W)_n \frac{< n >^s}{1 + i(n/k - t)}\right) \left(< n >^s (1 + i(n/k - t))\langle e^{iny}, R\rangle\right) \\
&\leq \left\|A_n(\partial_y W)_n \frac{< n >^s}{1 + i(n/k - t)}\right\|_{l^2_n} \left\|< n >^s (1 + i(n/k - t))\langle e^{iny}, R\rangle\right\|_{l^2_n}.
\end{aligned}$$

We, in particular, note that

$$\frac{1}{|1 + i(n/k - t)|^2} \in L^1_t.$$

Thus, it suffices to control

$$\sum_n < n >^{2s} |(1 + i(n/k - t))\langle e^{iny}, R\rangle|^2. \tag{129}$$

As

$$ine^{iny} = \partial_y e^{iny},$$

and as R has zero boundary values, integrating by parts

$$(1 + i(n/k - kt))\langle e^{iny}, R\rangle = \langle e^{iny}, R\rangle + \langle e^{iny}, (\frac{\partial_y}{k} - it)R\rangle.$$

By the triangle and Young's inequality, one thus obtains an estimate of (129) by

$$\|R\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)R\|_{H^s}^2.$$

Computing $(\frac{\partial_y}{k} - it)R$ by the product rule and using the triangle inequality then concludes the proof. $\square$

By Proposition 5.5 of Section 1, for f, g sufficiently regular, it hence suffices to estimate

$$\|\Phi\|_{H^s} + \|(\frac{\partial_y}{k} - it)\Phi\|_{H^s} + \|\Phi^{(1)}\|_{H^s} + \|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}.$$

As the estimates for Φ and $\Phi^{(1)}$ are very similar, to simplify notation and as we will later on also derive such an estimate for $\Phi^{(2)}$, we in the following consider a general problem:

Let ψ solve

$$\begin{aligned}(-1 + (g(\frac{\partial_y}{ik} - t))^2)\psi &= R,\\ \psi|_{y=0,1} &= 0,\\ y &\in [0,1]\\ C > g^2 > c > 0, g &\in W^{2,\infty}.\end{aligned} \tag{ELL}$$

for some $R \in H^s, 0 \leq s < 1/2$.

In the following we show that, as in the case $s = 0$, for $|k^{-1}|$ sufficiently small

$$\|\psi\|_{H^s}^2 + \|(\frac{\partial_y}{ik} - t)\psi\|_{H^s}^2 \lesssim \sum c_n(t) < n >^{2s} |R_n|^2,$$

for some family $c_n(t) \in L_t^1$ with $\|c_n(t)\|_{L_t^1} < C < \infty$ uniformly in n.

As in the case $s = 0$, the heuristic idea, is to consider the inner product (now in H^s) of the first equation in (ELL) with ψ and estimate:

(lower)

$$\|\psi\|_{H^s}^2 + \|(\frac{\partial_y}{ik} - t)\psi\|_{H^s}^2 \lesssim \Re\langle\psi, R\rangle_{H^s} - \text{errors},$$

(upper)

$$\Re\langle\psi, R\rangle_{H^s} \lesssim \left(\|\psi\|_{H^s}^2 + \|(\frac{\partial_y}{ik} - t)\psi\|_{H^s}^2\right)^{1/2} \left(\sum c_n(t) < n >^{2s} |R_n|^2\right)^{1/2}.$$

Here, errors are terms that can either be absorbed in the left-hand-side or estimated by terms similar to the right-hand-side in (upper).

As we work in fractional Sobolev spaces, integration by parts and similar estimates involve many more boundary terms, commutators and other corrections. Controlling all these terms in a suitable way, makes (lower) technically much more challenging than in the integer Sobolev case. The upper estimate, however, follows analogously, as is shown in the following lemma.

Lemma 5.3. *Let ψ, R solve* (ELL)*, then*

$$\Re\langle\psi, R\rangle_{H^s} \lesssim \left(\|\psi\|_{H^s}^2 + \|(\frac{\partial_y}{ik} - t)\psi\|_{H^s}^2\right)^{1/2} \left(\sum c_n(t) < n >^{2s} |R_n|^2\right)^{1/2},$$

where

$$c_n(t) = \frac{1}{1 + (\frac{n}{k} - t)^2} \in L_t^1.$$

Proof of Lemma 5.3. Following the same strategy as in Lemma 5.2, we express $\langle\psi, R\rangle_{H^s}$ in a basis, multiply by a factor

$$\frac{1 + i(n/k - t)}{1 + i(n/k - t)},$$

integrate by parts and employ Cauchy-Schwarz. $\square$

In order to derive (lower), we first make use of our freedom in choosing the error term, by modifying the (shifted) elliptic operator.

$$\begin{aligned}&(-1+(g(\frac{\partial_y}{ik}-t))^2)\psi\\&=-\psi+(\frac{\partial_y}{ik}-t)g^2(\frac{\partial_y}{ik}-t)\psi-\frac{g'}{ik}g(\frac{\partial_y}{ik}-t)\psi.\end{aligned}$$

Up to boundary terms, the leading operator

$$-1+(\frac{\partial_y}{ik}-t)g^2(\frac{\partial_y}{ik}-t)$$

is hence symmetric and negative definite, which we use for a lower estimate in Lemma 5.5 and in combination with Proposition 5.6.

LEMMA 5.4. *Let* $\psi\in H^s([0,1])$ *be a solution of* (ELL). *Then,*

$$\left|\left\langle\psi,\frac{g'}{ik}g\left(\frac{\partial_y}{ik}-t\right)\psi\right\rangle_{H^s}\right|\lesssim\frac{1}{|k|}\|\|\psi\|_{H^s}\|(\frac{\partial_y}{ik}-t)\psi\|_{H^s}^2.$$

For k sufficiently large, instead of (lower)*, it thus suffices to prove*

$$\|\psi\|_{H^s}^2+\|(\frac{\partial_y}{k}-it)\psi\|_{H^s}^2\lesssim\langle\psi,-\psi+(\frac{\partial_y}{ik}-t)g^2(\frac{\partial_y}{ik}-t)\psi\rangle_{H^s}-\textit{errors}.$$

PROOF OF LEMMA 5.4. The first statement follows by Cauchy-Schwarz and applying Proposition 5.5 of Section 1 with $gg'\in W^{1,\infty}(\mathbb{T})$.

For the second statement, we note that

$$\begin{aligned}c(\|\psi\|_{H^s}^2+\|(\frac{\partial_y}{k}-it)\psi\|_{H^s}^2)&\leq\Re\langle\psi,R\rangle-\text{errors}\\&=\Re\langle\psi,-\psi+(\frac{\partial_y}{ik}-t)g^2(\frac{\partial_y}{ik}-t)\psi\rangle_{H^s}-\text{errors}\\&\quad+\Re\langle\psi,\frac{g'}{ik}g(\frac{\partial_y}{ik}-t)\psi\rangle_{H^s}\\&\leq\Re\langle\psi,-\psi+(\frac{\partial_y}{ik}-t)g^2(\frac{\partial_y}{ik}-t)\psi\rangle_{H^s}-\text{errors}\\&\quad+\frac{C}{|k|}(\|\psi\|_{H^s}^2+\|(\frac{\partial_y}{k}-it)\psi\|_{H^s}^2).\end{aligned}$$

Letting $|k|\gg 0$ be sufficiently large, $\frac{C}{k}\leq c/2$, which allows us to absorb the last term in the left-hand-side. ☐

In order to prove (lower), it thus remains to show that

$$-\Re\langle\psi,(\frac{\partial_y}{ik}-t)g^2(\frac{\partial_y}{ik}-t)\psi\rangle_{H^s}$$

provides a control of

$$\|(\frac{\partial_y}{ik}-t)\psi\|_{H^s}^2,$$

up to error terms.

While in the case $s=0$ this reduces to an integration by parts argument, for $s>0$ two additional challenges arise:

- Integrating by parts yields boundary terms.
- $\langle u,g^2u\rangle_{H^s}\neq\langle gu,gu\rangle_{H^s}\not\gtrsim\min(g^2)\|u\|_{H^s}^2$.

The second issue is addressed by Proposition 5.6 in Section 1 and the former by the following two lemmata.

LEMMA 5.5. *Let $\psi \in H^s([0,1])$ be a solution of* (ELL). *Then*

$$\left|\left\langle \psi, \left(\frac{\partial_y}{ik}-t\right) g^2 \left(\frac{\partial_y}{ik}-t\right)\psi\right\rangle_{H^s} + \left\langle \left(\frac{\partial_y}{ik}-t\right)\psi, g^2\left(\frac{\partial_y}{ik}-t\right)\psi\right\rangle_{H^s}\right|$$

$$\lesssim |k^{-1}| \left(\|\psi\|_{H^s}^2 + \left\|\left(\frac{\partial_y}{ik}-t\right)\psi\right\|_{H^s}^2\right)^{1/2} \left\|\frac{< n >^s}{< n/k-t >}\right\|_{l^2} \left|g^2\left(\frac{\partial_y}{k}-it\right)\psi|_{y=0}^1\right|.$$

Furthermore,

$$\left\|\frac{< n >^s}{< n/k-t >}\right\|_{l^2} \lesssim_s < kt >^s .$$

PROOF. Expanding both terms in a Fourier basis and integrating by parts, the difference is given by

$$\sum_n < n >^{2s} \psi_n \frac{1}{k} g^2 (\frac{\partial_y}{k} - it)\psi\Big|_{y=0}^1 .$$

Taking absolute values inside the sum, multiplying by a factor

$$1 = \frac{1+i(n/k-t)}{1+i(n/k-t)}$$

and using Cauchy-Schwarz, the first estimate is proven.

For the second estimate, we note that

$$< n >^s \lesssim k^s < n/k-t >^s + < kt >^s,$$

and that

$$< n/k-t >^{s-1} \in l_n^2,$$

provided $s<1/2$.

□

LEMMA 5.6. *Let ψ, R solve* (ELL)*, then the following estimates hold:*

(a) $$\left|g^2\left(\frac{\partial_y}{k}-it\right)\psi|_{y=0}^1\right| \lesssim |k|^{-1} < t >^{-s} \left(\sum_n |R_n|^2 c_n(t) < n >^{2s}\right)^{1/2}.$$

(b) $$g^2\left(\frac{\partial_y}{ik}-t\right)\psi|_{y=0} = k\langle R, e^{ikty}u_1\rangle_{L^2},$$
$$g^2\left(\frac{\partial_y}{ik}-t\right)\psi|_{y=1} = k\langle R, e^{ikt(y-1)}u_2\rangle_{L^2},$$

(c) $$|\langle R, e^{ikty}u_1\rangle_{L^2}| \lesssim \sum_n |R_n| < \frac{n}{k}-t >^{-1},$$
$$|\langle R, e^{ikt(y-1)}u_2\rangle_{L^2}| \lesssim \sum_n |R_n| < \frac{n}{k}-t >^{-1},$$
$$|\langle R, e^{ikt(y-1)}u_2\rangle_{L^2} + \langle R, e^{ikty}u_1\rangle_{L^2}| \lesssim |k^{-1}| \sum_n |R_n| < \frac{n}{k}-t >^{-2} .$$

Proof of Lemma 5.6. We first show that that (b) and (c) imply (a). Thus, assume for the moment, that (c) holds. Then

$$
\begin{aligned}
\left| g^2(\frac{\partial_y}{k} - it)\psi|_{y=0}^1 \right| \lesssim & |k^{-1}| \sum_n |R_n| < \frac{n}{k} - t >^{-2} \\
= & |k^{-1}| \sum_n |R_n| \frac{< n >^s}{< n/k - t >^{1/2+\epsilon}} \frac{1}{< n/k - t >^{1/2+\epsilon}} \frac{1}{< n >^s < n/k - t >^{1-2\epsilon}} \\
\leq & |k^{-1}| \left(\sum_n |R_n|^2 c_n(t) < n >^{2s} \right)^{1/2} \\
& \left\| \frac{1}{< n/k - t >^{1/2+\epsilon}} \right\|_{l^2} \left\| \frac{1}{< n >^s < n/k - t >^{1-2\epsilon}} \right\|_{l^\infty},
\end{aligned}
$$

where

$$
c_n(t) = < n/k - t >^{-1-2\epsilon} \in L^1_t.
$$

We further estimate

$$
\begin{aligned}
\| \frac{1}{< n/k - t >^{1/2+\epsilon}} \|_{l^2} \lesssim \sqrt{k}, \\
\| \frac{1}{< n >^s < n/k - t >^{1-2\epsilon}} \|_{l^\infty} \leq < kt >^{-s} + < kt >^{-1+2\epsilon} .
\end{aligned}
$$

As $s < 1/2 < 1$, for $\epsilon > 0$ sufficiently small $1 - 2\epsilon \geq s$, which concludes the proof of (a).

The estimates (b) have been proven previously in Lemma 5.1 for the case of $\psi = \Phi$. Let again $e^{ikty} u_1$, $e^{ikt(y-1)} u_2$ be the homogeneous solutions with boundary values zero and one. Testing the equation and integrating by parts twice, yields two boundary terms. In the case of $e^{ikty} u_1$, the first boundary term is given by

$$
e^{ikty} u_1 \frac{1}{ik} g^2(\frac{\partial_y}{ik} - t)\psi|_{y=0}^1 = -\frac{1}{ik} g^2(\frac{\partial_y}{ik} - t)\psi|_{y=0},
$$

by the choice of the boundary values of $e^{ikty} u_1$. The second boundary term

$$
\psi \frac{1}{ik} g^2(\frac{\partial_y}{ik} - t) e^{ikty} u_1|_{y=0}^1,
$$

vanishes as ψ vanishes on the boundary. The result for $e^{ikt(y-1)} u_2$ follows analogously, which concludes the proof of (b).

It remains to prove (c). For the first two estimates, it suffices to prove that

$$
\begin{aligned}
\langle e^{iny}, e^{ikty} u_1 \rangle_{L^2} \lesssim < n/k - t >^{-1}, \\
\langle e^{iny}, e^{ikt(y-1)} u_2 \rangle_{L^2} \lesssim < n/k - t >^{-1} .
\end{aligned}
$$

A first, easy but non-optimal proof integrates $e^{i(kt-n)y}$ by parts, which yields a control by

$$
\left| \frac{k}{kt - n} \right| .
$$

For an improved estimate we recall that u_j is given by linear combinations of

$$
e^{\pm k U^{-1}(y)},
$$

and that

$$
e^{i(kt-n)y \pm k U^{-1}(y)} = \frac{1}{\pm k (U^{-1})' + i(kt - n)} \partial_y e^{i(kt-n)y \pm k U^{-1}(y)}.
$$

The improved final estimate of (c), follows by noting that $e^{ikt(y-1)}u_2 + e^{ikty}u_1$ has boundary values $1, 1$ and is thus periodic. A first integration by parts thus does not yield any boundary contribution and we may integrate by parts once more to obtain the quadratic decay.

$\square$

Combining both lemmata, we thus have further simplified (lower) to estimating

$$\left\langle \left(\frac{\partial_y}{ik} - t\right)\psi, g^2\left(\frac{\partial_y}{ik} - t\right)\psi \right\rangle_{H^s}.$$

Employing Proposition 5.6 of Section 1, as well as the L^2 estimate of Chapter 4, we have thus proven the following proposition:

PROPOSITION 5.7. *Let ψ, R solve* (ELL), $0 \leq s < 1/2$ *and* $R \in H^s$. *Then*

$$\|\psi\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\psi\|_{H^s} \lesssim \sum_n |R_n|^2 c_n(t) < n >^{2s},$$

where $c_n \in L^1_t$ with $\|c_n\|_{L^1_t}$ bounded uniformly in n.

Having derived this generic result for (ELL), it remains to apply it to the cases $\psi = \Phi$ and $\psi = \Phi^{(1)}$.

PROPOSITION 5.8. *Let $0 < s < 1/2$, $W \in H^s$ and let Φ be a solution of*

$$\begin{aligned}(-k^2 + (g(\partial_y - ikt))^2)\Phi &= W,\\ \Phi|_{y=0,1} &= 0,\\ y &\in [0,1].\end{aligned}$$

Let further $g, g' \in W^{1,\infty}(\mathbb{T})$ and $g^2 > c > 0$. Then there exists a constant such that

$$\|\Phi\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)^2\Phi\|_{H^s}^2 \lesssim \sum_n |W_n|^2 < n >^2 c_n(t),$$

for some $c_n(t) \in L^1_t$.

PROOF OF PROPOSITION 5.8. Applying Proposition 5.7 with $\psi = \Phi$, $R = W$ yields the result. $\square$

Considering the case $\psi = \Phi^{(1)}$, the upper estimate, Lemma 5.2, has to be slightly modified, as the second term in

$$R = \partial_y W + \left[(\frac{\partial_y}{k} - it)g^2(\frac{\partial_y}{k} - it), \partial_y\right]\Phi$$

has to be treated separately.

LEMMA 5.7. *Let Φ, W solve*

$$\begin{aligned}(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi &= W,\\ \Phi|_{y=0,1} &= 0,\\ y &\in [0,1].\end{aligned}$$

Then,

$$\Re\langle\Phi^{(1)}, [(\frac{\partial_y}{k} - it)g^2(\frac{\partial_y}{k} - it), \partial_y]\Phi\rangle_{H^s} \lesssim \sum_n < n >^{2s} c_n(t)(|(\partial_y W)_n|^2 + |W_n|^2).$$

Proof of Lemma 5.7. We compute

$$\left[(\frac{\partial_y}{k} - it)g^2(\frac{\partial_y}{k} - it), \partial_y\right]\Phi = 2(\frac{\partial_y}{k} - it)gg'(\frac{\partial_y}{k} - it)\Phi.$$

Integrating by parts, we thus obtain a bulk term

$$\langle(\frac{\partial_y}{k} - it)\Phi^{(1)}, 2gg'(\frac{\partial_y}{k} - it)\Phi\rangle_{H^s},$$

and, similar to Lemma 5.4, a boundary term

$$\sum_n \Phi_n^{(1)} < n >^{2s} k^{-1} 2gg'(\frac{\partial_y}{k} - it)\Phi. \tag{130}$$

Using Proposition 5.5 of Section 1 and Young's inequality, the bulk term can be estimated by

$$\epsilon\|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}^2 + \epsilon^{-1}C\|(\frac{\partial_y}{k} - it)\Phi\|_{H^s}^2.$$

Here, the second term can be estimated by the previous proposition, while the first term can be absorbed in the left-hand-side of the estimate as in the proof of Lemma 5.4.

In order to estimate the boundary term, (130), we follow the same strategy as in the proof of Lemma 5.5 and Lemma 5.6. We thus obtain an estimate by

$$\|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}\|\frac{< n >^s}{< n/k - t >}\|_{l^2}\left|2gg'(\frac{\partial_y}{k} - it)\Phi|_{y=0}^1\right|.$$

It remains to estimate

$$\left|2gg'(\frac{\partial_y}{k} - it)\Phi|_{y=0}^1\right|.$$

Unlike in the last case of (c) in Lemma 5.6, there is no additional cancellation of the contributions at $y = 0$ and $y = 1$. Hence, we estimate

$$|2gg'| \lesssim \|g\|_{W^{1,\infty}}^2$$

and consider the contributions at $y = 0$ and $y = 1$ separately. Using Lemma 5.6, we express

$$(\frac{\partial_y}{k} - it)\Phi|_{y=0,1}$$

in terms of

$$\langle W, e^{ikty}u_1\rangle_{L^2} = \frac{1}{ikt}W|_{y=0} + \frac{1}{ikt}\langle e^{ikty}\partial_y W u_1\rangle_{L^2}.$$

To estimate both terms, we follow the same strategy as in the proof of Theorem 5.2. The first term is controlled using Young's inequality, i.e.

$$\begin{aligned}&\|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}\|\frac{< n >^s}{< n/k - t >}\|_{l^2}\frac{W|_{y=0,1}}{ikt} \\ &\lesssim \epsilon^{-1}|kt|^{-2\gamma} + \epsilon\|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}^2\frac{< kt >^{2s}}{|kt|^{2(1-\gamma)}},\end{aligned}$$

where $\gamma > 1/2$ is chosen such that $1 - \gamma \geq s$. The first term is integrable in time and the second can be absorbed in the left-hand-side.

It remains to estimate

$$\|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}\|\frac{< n >^s}{< n/k - t >}\|_{l^2}\frac{1}{ikt}\langle e^{ikty}\partial_y W u_1\rangle_{L^2}. \tag{131}$$

For this purpose, we compute

$$\langle e^{ikty}, \partial_y W u_1\rangle_{L^2} = \langle e^{ikty}, u_1\partial_y W\rangle_{L^2} + \langle e^{ikty}, W\partial_y u_1\rangle_{L^2}.$$

The second term can be integrated by parts once more to obtain another factor $\frac{1}{ikt}$ and is thus easily controlled. For the first term we estimate

$$\langle e^{ikty}, u_1 \partial_y W \rangle_{L^2} \lesssim \sum |(\partial_y W)_n| \frac{< n >^s}{< n/k - t >^{1-\lambda}} \frac{1}{< n >^s < n/k - t >^{\lambda}},$$

where $0 < \lambda < 1$ and $s + \lambda > 1/2$.

The terms in (131) can thus be estimated by

$$\begin{aligned} &\|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s} < kt >^s \frac{1}{|kt|} \left\| |(\partial_y W)_n| \frac{< n >^s}{< n/k - t >^{1-\lambda}} \right\|_{l^2} \left\| \frac{1}{< n >^s < n/k - t >^{\lambda}} \right\|_{l^2} \\ \lesssim &\|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s} \frac{1}{|kt|} \left\| |(\partial_y W)_n| \frac{< n >^s}{< n/k - t >^{1-\lambda}} \right\|_{l^2}. \end{aligned}$$

Using Young's inequality, the first factor can be absorbed, while the second factor is of the desired form with

$$c_n(t) := \frac{1}{|kt|} \frac{1}{< n/k - t >^{2(1-\lambda)}} \in L^1_t.$$

□

This concludes the stability proof in $H^s, s < 3/2$.

As a consequence we now have sufficient control of regularity to obtain damping with integrable rates and scattering.

Corollary 5.3 (Scattering). *Let $0 < s < 1/2$ and let W be a solution of the linearized Euler equations,* (122)*, such that $\|\partial_y W\|_{H^s}$ and $\|W\|_{H^1}$ are uniformly bounded (e.g. satisfying Theorem 5.1). Then there exists $W^{\infty} \in H^s_y L^2_x$ such that, as $t \to \infty$,*

$$\begin{aligned} \|V_2\|_{L^2} &= \mathcal{O}(t^{-(1+s)}), \\ W &\xrightarrow{L^2} W_{\infty}, \\ \|W(t) - W_{\infty}\|_{L^2} &= \mathcal{O}(t^{-s}). \end{aligned}$$

Proof of Corollary 5.3. This result is proven in the same way as Theorem 4.16. That is, by Duhamel's formula, $W(t)$ satisfies

$$W(t) = \omega_0 + \int_0^t f V_2(\tau) d\tau.$$

Estimating and integrating,

$$\|f V_2(\tau)\|_{L^2} \leq \|f\|_{L^{\infty}} \|V_2\|_{L^2} = \mathcal{O}(t^{-(1+s)}),$$

then yields the result. □

Approximating $\omega_0 \in L^2$ by functions in $H^s, 1 < s < 3/2$, we obtain scattering in L^2.

Corollary 5.4 (L^2 scattering). *Let f, g, k be as in Theorem 5.1. Then for any $\omega_0 \in L^2$ there exists $W_{\infty} \in L^2$ such that*

$$W \xrightarrow{L^2} W_{\infty},$$

as $t \to \infty$.

Proof of Corollary 5.4. Let $(\omega_0^n)_{n \in \mathbb{N}} \in H^s$ be a sequence such that

$$\omega_0^n \xrightarrow{L^2} \omega_0,$$

as $n \to \infty$. By Corollary 5.4, for any ω_0^n there exists an asymptotic profile W_{∞}^n. By the L^2 stability Theorem 4.10 the convergence of ω_0^n also implies the convergence of

$W^n(t)$ at any time t and of W^n_∞. The result then follows by choosing an appropriate diagonal sequence in t and n. □

3. Stability in $H^{5/2-}$

In the previous Section 2, we have seen that, under general perturbations, the critical Sobolev exponent in y is given by $s = \frac{3}{2}$. More precisely, for any $m \in \mathbb{N}_0$, we have shown stability in the periodic fractional Sobolev spaces

$$H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T}), s < \frac{3}{2},$$

and that stability in

$$H^m_x H^s_y(\mathbb{T}_L \times [0,1]), s > \frac{3}{2},$$

can in general not hold, unless one restricts to initial perturbations ω_0 with zero Dirichlet boundary data, $\omega_0|_{y=0,1} = 0$.

Restricting to such perturbations, Theorem 4.15 of Chapter 4 establishes stability in $H^m_x H^2_y(\mathbb{T}_L \times [0,1])$, which is sufficient to prove linear inviscid damping with the optimal algebraic rates. However, as discussed in Section 4 of Chapter 4, H^2 stability is not sufficient to establish consistency with the nonlinear equations. As the main result of this section, we hence show that, for this restricted class of perturbations, ω_0, the critical Sobolev exponent in y is given by $s = \frac{5}{2}$. More precisely, as shown in Corollary 5.5, for initial perturbations, ω_0, with zero Dirichlet data, $\omega_0|_{y=0,1} = 0$, generically $\partial_y^2 W$ asymptotically develops (logarithmic) singularities a the boundary. Hence, even for this restricted class of perturbations, stability in $H^m_x H^s_y(\mathbb{T}_L \times [0,1]), s > \frac{5}{2}$, can in general not hold. As we discuss in Chapter 6, this further implies instability of the nonlinear problem in the finite periodic channel in high Sobolev spaces and therefore, in particular, forbids nonlinear inviscid damping results in Gevrey regularity such as in the work of Bedrossian and Masmoudi, [**BM13b**].

As a complementary result to the instability, Theorem 5.4 establishes stability in the periodic fractional Sobolev spaces, $H^m_x H^s_y(\mathbb{T}_L \times \mathbb{T}), s < \frac{5}{2}$. As remarked in Section 4 of Chapter 4, this additional stability allows us to prove consistency with the nonlinear problem, also for the finite periodic channel.

We recall that the linearized Euler equations, (122), decouple with respect to k and we may hence consider k as a given parameter and consider the stability of

$$W(t) = W(t, k, \cdot) \in H^s([0,1]) \text{ or } H^s(\mathbb{T}).$$

The following two lemmata provide a description of the evolution of derivatives of Φ on the boundary. Using these lemmata, in Proposition 5.5 we show that, in general, stability in $H^s([0,1]), s > \frac{5}{2}$, can not hold.

Lemma 5.8. *Let W be a solution of the linearized Euler equations,* (122)*, and suppose that $\|W\|_{H^2([0,1])}$ is bounded uniformly in time. Suppose further that $\omega_0|_{y=0,1} \equiv 0$. Then there exist constants $c_0, c_1 \in \mathbb{C}$ such that*

$$\partial_y W|_{y=0} \to c_0,$$
$$\partial_y W|_{y=1} \to c_1,$$

as $t \to \infty$.

We remark that c_0, c_1 are in general non-trivial and not determined by $\partial_y \omega_0|_{y=0,1}$. In analogy to Corollary 5.1, in Corollary 5.5 we show that non-trivial c_0, c_1 asymptotically result in a (logarithmic) blow-up at the boundary and thus provide an upper limit on stability results.

PROOF OF LEMMA 5.8. Restricting the evolution equation for $\partial_y W$, (123), to the boundary, we obtain

$$\partial_t \partial_y W|_{y=0,1} = \frac{if}{k} \partial_y \Phi|_{y=0,1},$$

where we used that $\Phi|_{y=0,1} \equiv 0$. It therefore suffices to show that $\partial_y \Phi|_{y=0,1}$ decays in t at an integrable rate. We recall that by Lemma 5.6

$$\begin{aligned} \partial_y \Phi|_{y=0} &= \frac{k}{g(0)} \langle W, e^{ikty} u_1(0,y) \rangle_{L^2}, \\ \partial_y \Phi|_{y=1} &= \frac{k}{g(1)} \langle W, e^{ikt(y-1)} u_2(0,y) \rangle_{L^2}. \end{aligned}$$

As $k \neq 0$ and as g is bounded away from 0, it suffices to consider the L^2 products. Integrating by parts once, we obtain

$$\begin{aligned} \langle W, e^{ikty} u_1(0,y) \rangle_{L^2} &= -\frac{1}{ikt} W|_{y=0} - \frac{1}{ikt} \langle e^{ikty}, \partial_y (W u_1(0,y)) \rangle_{L^2} \\ &= -\frac{1}{ikt} \langle e^{ikty}, \partial_y (W u_1(0,y)) \rangle_{L^2}. \end{aligned}$$

Recalling Lemma 5.1, a uniform control of $\|W\|_{H^s} + \|\partial_y W\|_{H^s}$ for some $s > 0$ suffices to obtain an upper bound by $\mathcal{O}(t^{-1-s})$ and thus deduce the result.

Integrating by parts once more, we obtain

$$\langle W, e^{ikty} u_1(0,y) \rangle_{L^2} = \frac{1}{k^2 t^2} e^{ikty} \partial_y (W u_1(0,y))|_{y=0}^1 - \frac{1}{k^2 t^2} \langle e^{ikty}, \partial_y^2 (W u_1(0,y)) \rangle_{L^2}.$$

Again using the assumption that $W|_{y=0,1} \equiv 0$, the first term can be controlled by

$$C_k t^{-2} |\partial_y W|_{y=0,1}|,$$

and the second term by

$$C_k t^{-2} \|W\|_{H^2}^2.$$

Using the uniform control of $\|W\|_{H^2}$, we thus obtain the differential inequality

$$|\partial_t \partial_y W|_{y=0,1}| \lesssim t^{-2} (|\partial_y W|_{y=0,1}| + 1).$$

Integrating this inequality then yields the result. □

Following a similar approach as in Section 2, we show that $\partial_y^2 W|_{y=0,1}$ in general grows unboundedly as $t \to \infty$.

LEMMA 5.9. *Let W be a solution of the linearized Euler equations,* (122), *and suppose that, for some $s > 0$, $\|W\|_{H^2}$ and $\|\partial_y^2 W\|_{H^s}$ are bounded uniformly in time. Then,*

$$\partial_y^2 \Phi|_{y=0,1} = \frac{1}{ikt} \partial_y W|_{y=0,1} + \mathcal{O}(t^{-1-s}).$$

PROOF. Following the same approach as in the proof of Lemma 4.10, we note that by (122),

$$(-1 + (g(y)(\frac{\partial_y}{k} - it))^2)\Phi = W,$$

and by the choice of zero Dirichlet boundary values of Φ and W,

$$g^2 \partial_y^2 \Phi|_{y=0,1} = (-gg' + iktg^2) \partial_y \Phi|_{y=0,1}.$$

Dividing by g^2 and using

$$\begin{aligned}\partial_y \Phi|_{y=0} &= \frac{k}{g^2(0)} \langle W, e^{ikty} u_1 \rangle \\ &= \frac{k}{g^2(0)} \left(\frac{1}{ikt} \omega_0|_{y=0} + \langle e^{ikty}, \partial_y W u_1 \rangle \right), \\ \partial_y \Phi|_{y=1} &= \frac{k}{g^2(1)} \langle W, e^{ikt(y-1)} u_2 \rangle \\ &= \frac{k}{g^2(1)} \left(\frac{1}{ikt} \omega_0|_{y=1} + \langle e^{ikt(y-1)}, \partial_y W u_2 \rangle \right),\end{aligned}$$

from Lemma 5.1, it thus suffices to consider

$$\langle e^{ikty}, \partial_y W u_1 \rangle,$$
$$\langle e^{ikt(y-1)}, \partial_y W u_2 \rangle.$$

Integrating e^{ikty} or $e^{ikt(y-1)}$ by parts and using boundary values of u_1, u_2, yields the leading terms

$$\frac{1}{ikt} \partial_y W|_{y=0,1},$$

while the remainder is given by

$$\frac{1}{ikt} \langle e^{ikty}, \partial_y(\partial_y W u_1) \rangle,$$
$$\frac{1}{ikt} \langle e^{ikt(y-1)}, \partial_y(\partial_y W u_2) \rangle,$$

respectively. By the product rule

$$\partial_y(\partial_y W u_j) = u_j \partial_y^2 W + \partial_y W \partial_y u_j.$$

For the latter term integrating by parts once more yields a term controlled by

$$\mathcal{O}((kt)^{-2}) \|W\|_{H^2}.$$

It thus suffices to consider only

$$\frac{1}{ikt} \langle e^{ikty} u_1, \partial_y^2 W \rangle,$$
$$\frac{1}{ikt} \langle e^{ikt(y-1)} u_2, \partial_y^2 W \rangle.$$

Expanding into a basis and using duality, the result then follows by estimating

$$\|e^{ikty} u_1\|_{H^{-s}} + \|e^{ikt(y-1)} u_2\|_{H^{-s}} = \mathcal{O}(t^{-s}).$$

□

Corollary 5.5. *Let $\omega_0|_{y=0,1} \equiv 0$ and let W be the solution of* (122). *Further suppose that the limits*

$$\lim_{t \to \infty} f(y) \partial_y W|_{y=0,1}$$

exist (e.g. by Lemma 5.8) and are non-trivial. Then for any $s > 5/2$,

$$\sup_{t \geq 0} \|W\|_{H^s} = \infty.$$

Proof. Suppose to the contrary that for some $s > 5/2$, $\|W\|_{H^s}$ is bounded uniformly in time. Then, by Lemma 5.8,

$$\partial_t \partial_y^2 W|_{y=0,1} = \frac{if}{k} \partial_y^2 \Phi + \frac{if'}{k} \partial_y \Phi|_{y=0,1} = \frac{if}{k^2 t} \partial_y W|_{y=0,1} + \mathcal{O}(t^{-1-s}).$$

Integrating this equation, we thus obtain that

$$\log(t) \lesssim |\partial_y^2 W|_{y=0,1}| \leq \|\partial_y^2 W\|_{L^\infty},$$

as $t \to \infty$. On the other hand by the Sobolev embedding and the choice of $s > \frac{5}{2}$,

$$\|\partial_y^2 W\|_{L^\infty} \lesssim \|W\|_{H^s},$$

which we supposed to be bounded uniformly in time. This hence yields a contradiction, which proves the desired result. □

The main result of this section is given by the following Theorem 5.4, which proves that the above restriction is sharp in the sense that stability holds for $s < 5/2$. More precisely, as in Section 2, instead of $H^s([0,1])$, we consider *periodic* spaces, i.e.

$$W(t,k,\cdot) \in H^{s-1}(\mathbb{T}), \partial_y W(t,k,\cdot) \in H^{s-1}(\mathbb{T}),$$

which allows us to use both a Fourier characterization and a kernel characterization.

THEOREM 5.4. *Let $0 < s < 1/2$ and let $\omega_0 \in H^2([0,1])$, with vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$, and $\omega_0, \partial_y\omega_0, \partial_y^2\omega_0 \in H^s(\mathbb{T})$. Suppose further that $f, g \in W^{3,\infty}(\mathbb{T})$ satisfy the assumptions of the H^2 stability result, Theorem 4.15, and that*

$$\|f\|_{W^{3,\infty}(\mathbb{T})}L$$

is sufficiently small. Then the solution, W, of the linearized Euler equations, (122), *satisfies*

$$\|\partial_y^2 W(t)\|_{H^s(\mathbb{T})} \lesssim \|\omega_0\|_{H^s} + \|\partial_y\omega_0\|_{H^s} + \|\partial_y^2\omega_0\|_{H^s},$$

uniformly in time.

REMARK 11. *Similar to Theorem 5.1, the assumptions on f and g are chosen such that we can apply Proposition 5.5 to the functions f, g and their derivatives f', f'' and g', g''. Furthermore, we require*

$$g^2 = U'(U^{-1}(\cdot))^2$$

to be such that we can apply Proposition 5.6.

As discussed in Remark 9, these assumptions can probably be relaxed to requiring that

$$f, g \in W^{4,\infty}([0,1]),$$

and that

$$|g^2(1) - g^2(0)| = |(U'(b))^2 - (U'(a))^2|$$

is sufficiently small compared to

$$\min(g^2) = \min((U')^2) > 0.$$

As in the previous section and as in Chapter 4, we split the contributions in the evolution equation into boundary corrections and potentials with zero Dirichlet conditions. Let thus W be a solution of (91), then $\partial_y^2 W$ satisfies (90):

$$\begin{aligned}
&\partial_t\partial_y^2 W = \frac{if}{k}(\Phi^{(2)} + H^{(2)}) + \frac{2f'}{ik}(\Phi^{(1)} + H^{(1)}) + \frac{f''}{ik}\Phi, \\
(132) \quad &(-1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(2)} = \partial_y^2 W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi, \\
&\Phi^{(2)}_{y=0,\pi} = 0,
\end{aligned}$$

and the *homogeneous correction*, $H^{(2)}$, satisfies

$$(-1+(g(\frac{\partial_y}{k}-it))^2)H^{(2)}=0,$$
$$H^{(2)}|_{y=0,\pi}=\partial_y^2\Phi|_{y=0,\pi}.$$

Furthermore, as discussed in the beginning of Section 2, $\Phi^{(1)}$ and $H^{(1)}$ satisfy (123):

$$\begin{aligned}\partial_t\partial_y W &= \frac{if}{k}\partial_y\Phi+\frac{if'}{k}\Phi,\\ (-1+(g(\frac{\partial_y}{k}-it))^2)\Phi^{(1)} &= \partial_y W+[(g(\partial_y-it))^2,\partial_y]\Phi,\\ \Phi^{(1)}_{y=0,\pi} &= 0,\\ H^{(1)} &= \partial_y\Phi-\Phi^{(1)},\\ (t,k,y) &\in \mathbb{R}\times L(\mathbb{Z}\setminus\{0\})\times[0,1],\end{aligned} \tag{133}$$

Considering a decreasing weight A and computing

$$\partial_t(\langle W,AW\rangle_{H^s}+\langle\partial_y W,A\partial_y W\rangle_{H^s}+\langle\partial_y^2 W,A\partial_y^2 W\rangle_{H^s})=:\partial_t I(t),$$

we show that $I(t)$ is uniformly bounded.

As we have seen in the previous Section 2, the first two terms are non-positive under the conditions of the theorem.

It thus remains to control

$$\partial_t\langle\partial_y^2 W,A\partial_y^2 W\rangle_{H^s}.$$

For this purpose, we have to estimate

$$\langle\frac{if''}{k}\Phi,A\partial_y^2 W\rangle_{H^s}+\langle\frac{if'}{k}\Phi^{(1)},A\partial_y^2 W\rangle_{H^s}+\langle\frac{if'}{k}\Phi^{(2)},A\partial_y^2 W\rangle_{H^s} \tag{elliptic}$$

$$+\langle\frac{if'}{k}H^{(1)},A\partial_y^2 W\rangle_{H^s}+\langle\frac{if}{k}H^{(2)},A\partial_y^2 W\rangle_{H^s} \tag{boundary}$$

in terms of

$$\frac{C}{|k|}|\langle W,\dot{A}W\rangle_{H^s}+\langle\partial_y W,\dot{A}\partial_y W\rangle_{H^s}+\langle\partial_y^2 W,\dot{A}\partial_y^2 W\rangle_{H^s}|.$$

Requiring $|k|\gg 0$ to be sufficiently large and thus $\frac{C}{|k|}$ to be sufficiently small, then yields the result. As in Section 2, the control of the boundary and elliptic contributions is obtained in the following subsections.

3.1. Boundary corrections. The following two theorems provide a control of the boundary contributions in the proof of Theorem 5.4. Here, Theorem 5.5 controls contributions by $H^{(1)}$ and Theorem 5.6 controls contributions by $H^{(2)}$, respectively.

THEOREM 5.5. *Let $0<s<1/2$ and let W,f,g as in Theorem 5.4. Let further A be a diagonal operator comparable to the identity, i.e.*

$$A: e^{iny}\mapsto A_n e^{iny},$$

with

$$1\lesssim A_n\lesssim 1,$$

uniformly in n. Then,

$$|\langle A\partial_y^2 W,\frac{if'}{k}H^{(1)}\rangle_{H^s}|\lesssim\sum_n c_n(t)<n>^{2s}(|(\partial_y^2 W)_n|^2+|(\partial_y W)_n|^2+|W_n|^2),$$

where $c_n\in L^1_t$ and $\|c_n\|_{L^1_t}$ is bounded uniformly in n.

Proof. Combining the approach of Lemma 4.9 and Theorem 5.2, we expand

$$H^{(1)} = \partial_y \Phi|_{y=0} e^{ikty} u_1 + \partial_y \Phi|_{y=1} e^{ikty(y-1)} u_2.$$

We may then estimate

$$|\langle A\partial_y^2 W, e^{ikty} u_1\rangle_{H^s}| \lesssim \sum_n < n >^{2s} |(\partial_y^2 W)_n| \frac{1}{|k| < n/k - t >}.$$

As by our assumptions $\omega_0|_{y=0,1} = 0$,

$$\partial_y \Phi|_{y=0} = \frac{k}{g^2(0)} \langle W, e^{ikty} u_1\rangle_{L^2}$$

has good decay in time. More precisely, as in Corollary 5.5, we integrate by parts twice to obtain control by

$$|\partial_y \Phi|_{y=0,1}| = \mathcal{O}(< kt >^{-2}) \|W\|_{H^2}.$$

Using the H^2 stability result of Chapter 4, we may thus estimate

$$|\langle A\partial_y^2 W, e^{ikty} u_1\rangle_{H^s}| \lesssim < kt >^{-2} \left\| \frac{< n >^s}{< n/k - t >^{(1-\gamma)}} (\partial_y^2 W)_n \right\|_{l^2_n} \left\| \frac{< n >^s}{< n/k - t >^{\gamma}} \right\|_{l^2}.$$

Choosing $0 < \gamma < 1$ sufficiently close to 1 such that $s - \gamma < -\frac{1}{2}$, then yields

$$\left\| \frac{< \eta >^s}{< n/k - t >^{\gamma}} \right\|_{l^2_n} = \mathcal{O}(< kt >^s).$$

The result thus follows with

$$c_n(t) := < kt >^{-2+s} < n/k - t >^{-2(1-\gamma)} \in L^1_t.$$

□

Theorem 5.6. *Let $0 < s < 1/2$ and let A, W, f, g as in Theorem 5.5. Then,*

$$|\langle A\partial_y^2 W, \frac{if}{k} H^{(2)}\rangle_{H^s}| \lesssim \sum_n c_n(t) < n >^{2s} (|(\partial_y^2 W)_n|^2 + |(\partial_y W)_n|^2 + |W_n|^2),$$

where $c_n \in L^1_t$ and $\|c_n\|_{L^1_t}$ is bounded uniformly in n.

Proof. Following the same approach as in Theorem 5.2 and Lemma 4.10, the estimate of

$$|\langle A\partial_y^2 W, \frac{if}{k} e^{ity} u_1\rangle_{H^s} \frac{1}{t} \langle \partial_y^2 W, e^{ity} u_1\rangle_{L^2}|$$

is identical up to a change of notation.

The additional boundary correction in the current case is given by

$$\frac{1}{ikt} \partial_y W|_{y=0,1}.$$

While $\partial_y W|_{y=0,1}$ is not conserved, by Lemma 5.8 it converges as $t \to \infty$ and is thus in particular bounded. This part of the estimate thus also concludes analogously to the proof of Theorem 5.2. □

3.2. Elliptic regularity. This subsection's main result is given by the following theorem, which provides control of the elliptic contributions in the proof of Theorem 5.4.

THEOREM 5.7. *Let $0 < s < 1/2$ and let A, f, g, W be a as in Theorem 5.5. Then*

$$\begin{aligned}&|\langle A\partial_y W, if\Phi^{(2)} + if'\Phi^{(1)} + if''\Phi^{(1)}\rangle_{H^s}|\\ &\lesssim \sum_n c_n(t) < n >^{2s} (|(\partial_y^2 W)_n|^2 + |(\partial_y W)_n|^2 + |W_n|^2),\end{aligned}$$

where $c_n \in L^1_t$ with $\|c_n\|_{L^1_t}$ bounded uniformly in n.

As in Section 2.2, Lemma 5.10 serves to reduce the proof of Theorem 5.4 to a fractional elliptic regularity problem. The desired elliptic estimate is then formulated in Proposition 5.9, whose prove is further broken down in Lemma 5.11 Lemma and 5.12.

LEMMA 5.10. *Let $0 < s < 1/2$ and let A, f, g, W as in Theorem 5.5. Then*

$$\begin{aligned}|\langle A\partial_y W, if\Phi^{(2)} + if'\Phi^{(1)} + if''\Phi^{(1)}\rangle_{H^s}| \lesssim &\left(\sum_n c_n(t) < n >^{2s} |(\partial_y^2 W)_n|^2\right)^{1/2}\\ &\big(\|if\Phi^{(2)}\|^2_{H^s} + \|(\frac{\partial_y}{k} - t)if\Phi^{(2)}\|^2_{H^s}\\ &+\|if'\Phi^{(1)}\|^2_{H^s} + \|(\frac{\partial_y}{k} - t)if'\Phi^{(1)}\|^2_{H^s}\\ &+\|if''\Phi\|^2_{H^s} + \|(\frac{\partial_y}{k} - t)if''\Phi\|^2_{H^s}\big)^{1/2}.\end{aligned}$$

PROOF. This result is proven in the same way as Lemma 5.2 in Section 2.2. □

The control of

$$\|if'\Phi^{(1)}\|^2_{H^s} + \|(\frac{\partial_y}{k} - t)if'\Phi^{(1)}\|^2_{H^s} + \|if''\Phi\|^2_{H^s} + \|(\frac{\partial_y}{k} - t)if''\Phi\|^2_{H^s}$$

by

$$\sum_n c_n(t) < n >^{2s} (|(\partial_y W)_n|^2 + |W_n|^2)$$

has already been obtained in the previous Section 2. It thus only remains to control

$$\|if\Phi^{(2)}\|^2_{H^s} + \|(\frac{\partial_y}{k} - t)if\Phi^{(2)}\|^2_{H^s},$$

which is formulated as the following proposition.

PROPOSITION 5.9. *Let f, g, ω_0, W be as in Theorem 5.4. Then,*

$$\|\Phi^{(2)}\|^2_{H^s} + \|(\frac{\partial_y}{k} - t)\Phi^{(2)}\|^2_{H^s} \lesssim \sum_n c_n(t) < n >^{2s} (|(\partial_y^2 W)_n|^2 + |(\partial_y W)_n|^2 + |W_n|^2).$$

PROOF OF PROPOSITION 5.9. We recall that $\Phi^{(2)}$ satisfies (132):

$$\begin{aligned}&(1 + (g(\frac{\partial_y}{k} - it))^2)\Phi^{(2)} = \partial_y^2 W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi,\\ &\Phi^{(2)}|_{y=0,1} = 0.\end{aligned}$$

Using the generic results of Section 2.2 with

$$\begin{aligned}&\psi = \Phi^{(2)},\\ &R = \partial_y^2 W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi,\end{aligned}$$

the result follows if we can obtain a good control of

$$\langle \psi, R\rangle_{H^s}$$

for our specific choice of R.

We note that

$$\langle \psi, \partial_y^2 W\rangle_{H^s} \lesssim (\|\psi\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\psi\|_{H^s}^2)^{1/2} \left(\sum_n < n/k - t >^{-2} < n >^{2s} (|(\partial_y^2 W)_n|^2\right)^{1/2},$$

is already of the desired form.

It thus remains to consider the commutator:

$$\begin{aligned}
&[(g(\frac{\partial_y}{k} - it))^2, \partial_y^2]\Phi \\
&=: (\frac{\partial_y}{k} - it)2gg'(\frac{\partial_y}{k} - it)\Phi^{(1)} + (\frac{\partial_y}{k} - it)(g^2)''(\frac{\partial_y}{k} - it)\Phi \\
&\quad + (\frac{\partial_y}{k} - it)2gg'(\frac{\partial_y}{k} - it)H^{(1)} + h(\frac{\partial_y}{k} - it)H^{(1)} \\
&\quad + \mathrm{Q},
\end{aligned}$$

where h can be computed in terms of the derivatives of g and Q is composed of terms involving only

$$\Phi, \Phi^{(1)}, (\frac{\partial_y}{k} - it)\Phi, (\frac{\partial_y}{k} - it)\Phi^{(1)}.$$

Thus,

$$\langle \psi, Q\rangle_{H^s} \lesssim \|\psi\|_{H^s}(\|\Phi\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{H^s}^2 + \|\Phi^{(1)}\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}^2)^{1/2},$$

which, by the $H^{3/2-}$ result, Theorem 5.1, can absorbed

$$\langle W, \dot{A}W\rangle_{H^s} + \langle \partial_y W, \dot{A}\partial_y W\rangle_{H^s} \leq 0.$$

The control of the remaining terms is obtained in the following two lemmata. $\square$

LEMMA 5.11. *Let g, ω_0, W be as in Theorem 5.4. Then,*

$$\begin{aligned}
&\langle \Phi^{(2)}, (\frac{\partial_y}{k} - it)2gg'(\frac{\partial_y}{k} - it)\Phi^{(1)} + (\frac{\partial_y}{k} - it)(g^2)''(\frac{\partial_y}{k} - it)\Phi\rangle_{H^s} \\
&\lesssim (\|\Phi^{(2)}\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\Phi^{(2)}\|_{H^s}^2)^{1/2} \\
&\quad \cdot \left(\|\Phi^{(1)}\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\Phi^{(1)}\|_{H^s}^2 + \|\Phi\|_{H^s}^2 + \|(\frac{\partial_y}{k} - it)\Phi\|_{H^s}^2\right)^{1/2} \\
&\quad + \sum_n c_n(t) < n >^{2s} (|(\partial_y^2 W)_n|^2 + |(\partial_y W)_n|^2 + |W_n|^2),
\end{aligned}$$

where $c_n \in L^1_t$ and $\|c_n\|_{L^1_t}$ is bounded uniformly in n.

PROOF OF LEMMA 5.11. Integrating the leading $(\frac{\partial_y}{k} - it)$ operators by parts, we obtain bulk terms

$$\langle (\frac{\partial_y}{k} - it)\Phi^{(2)}, 2gg'(\frac{\partial_y}{k} - it)\Phi^{(1)} + (g^2)''(\frac{\partial_y}{k} - it)\Phi\rangle_{H^s},$$

which can be controlled in the desired manner using Cauchy-Schwarz and Proposition 5.5 of Section 1.

It thus only remains to control the boundary contributions

$$\sum_n \Phi_n^{(2)} < n >^{2s} \left(2gg'(\frac{\partial_y}{k} - it)\Phi^{(1)} + (g^2)''(\frac{\partial_y}{k} - it)\Phi\right)\Big|_{y=0}^{1}.$$

Here we again estimate

$$\sum_n \Phi^{(2)}_n < n^{2s} > \lesssim (\|\Phi^{(2)}\|^2_{H^s} + \|(\frac{\partial_y}{k} - it)\Phi^{(2)}\|^2_{H^s})^{1/2} \| \frac{< n >^s}{< n/k - t >} \|_{l^2_n},$$

and

$$\| \frac{< n >^s}{< n/k - t >} \|_{l^2_n} \lesssim < kt >^s .$$

It remains to estimate

$$\big(2gg'(\frac{\partial_y}{k} - it)\Phi^{(1)} + (g^2)''(\frac{\partial_y}{k} - it)\Phi\big)|^1_{y=0},$$

where we may drop the terms which involve it, since Φ and $\Phi^{(1)}$ satisfy zero Dirichlet boundary conditions.

A good control of $\partial_y \Phi|_{y=0,1}$ in terms of $\|W\|_{H^2}$ has already been obtained in the proof of Corollary 5.2.

It thus remains to control $\partial_y \Phi^{(1)}|_{y=0,1}$. Following a similar approach as in Lemma 5.6, $\partial_y \Phi^{(1)}|_{y=0,1}$ can be computed by testing the right-hand-side of the equation against homogeneous solutions:

$$\langle \partial_y W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y]\Phi, e^{ikty} u_1 \rangle_{L^2},$$

$$\langle \partial_y W + [(g(\frac{\partial_y}{k} - it))^2, \partial_y]\Phi, e^{ikt(y-1)} u_2 \rangle_{L^2}.$$

In the case of the commutator terms, using integration by parts and the control of

$$\|(\frac{\partial_y}{k} - it)e^{ikty}u_1\|_{L^2},$$

we estimate by

$$\|\Phi\|_{L^2} + \|(\frac{\partial_y}{k} - it)\Phi\|_{L^2},$$

which is controlled. In order to estimate the remaining terms involving $\partial_y W$, we can either use the same approach as in Section 2.1 and control by

$$\sum_n c_n(t) < n >^{2s} |(\partial_y W)_n|^2,$$

or integrate e^{ikty} by parts to obtain an additional factor $\frac{1}{ikt}$ and estimate by

$$\frac{1}{t} \sum_n c_n(t) < n >^{2s} |(\partial_y^2 W)_n|^2.$$

$\square$

LEMMA 5.12. *Let g, ω_0, W be as in Theorem 5.4 and let $h \in W^{1,\infty}(\mathbb{T})$. Then,*

$$\begin{aligned}&|\langle \Phi^{(2)}, (\frac{\partial_y}{k} - it)2gg'(\frac{\partial_y}{k} - it)H^{(1)} + h(\frac{\partial_y}{k} - it)H^{(1)} \rangle_{H^s}| \\ \lesssim& \frac{1}{|k|}\left(\|\Phi^{(2)}\|_{H^s} + \|(\frac{\partial_y}{k} - it)\Phi^{(2)}\|_{H^s}\right) \frac{1}{t^{2-s}} \|W\|_{H^2}.\end{aligned}$$

PROOF OF LEMMA 5.12. Using the fact that $H^{(1)}$ solves

$$(-1 + (g(\frac{\partial_y}{k} - it))^2)H^{(1)} = 0,$$

as well as commuting some derivatives, one can express

$$\left(\frac{\partial_y}{k} - it\right) 2gg' \left(\frac{\partial_y}{k} - it\right) H^{(1)}$$

as

$$\left(\frac{\partial_y}{k} - it\right) h_1 H^{(1)} + h_2 H^{(1)},$$

for some functions $h_1, h_2 \in W^{1,\infty}(\mathbb{T})$.

Integrating the $(\frac{\partial_y}{k} - it)$ by parts and using Proposition 5.5, the bulk term is estimated by

$$\left(\|\Phi^{(2)}\|_{H^s} + \|(\frac{\partial_y}{k} - it)\Phi^{(2)}\|_{H^s}\right) \|H^1\|_{H^s},$$

while the boundary term is estimated in similar way as in the proof of Proposition 5.6, by

$$(\|\Phi^{(2)}\|_{H^s} + \|(\frac{\partial_y}{k} - it)\Phi^{(2)}\|_{H^s}) t^s |H^1|_{y=0,1}|.$$

As shown in the proof of Corollary 5.5:

$$|H^1|_{y=0,1}| = \mathcal{O}(t^{-2})\|W\|_{H^2}.$$

Furthermore,

$$H^{(1)} = H^{(1)}|_{y=0} e^{ikty} u_1 + H^{(1)}|_{y=1} e^{ikt(y-1)} u_2,$$

and

$$\|e^{ikty} u_1\|_{H^s}^2 \lesssim \sum_n \frac{< n >^{2s}}{< n - kt >^2} \lesssim t^{2s}.$$

Thus,

$$t^s |H^{(1)}|_{y=0,1}| + \|H^{(1)}\|_{H^s} \lesssim < t >^{s-2} \|W\|_{H^2},$$

which concludes the proof. □

We remark that under the conditions of Theorem 5.4, also stability in H^2 holds, as proven in Section 3.3 of Chapter 4. Thus, $\|W\|_{H^2}$ can be considered as a given constant. This then concludes the proof of Theorem 5.4.

Using these improved stability results, in the following Chapter 6, we revisit the problem of consistency and further consider the implications of these sharp (in)stability results for the nonlinear dynamics and for spectral stability as in Rayleigh's theorem.

Before that, however, in the following section, we further study the formation of singularities at the boundary, the behavior of the homogeneous corrections close the boundary and implications for (in)stability in fractional Sobolev spaces $W^{1,p}([0,1])$.

4. Boundary layers

Thus far we have seen that $\partial_y W$ and $\partial_y^2 W$, when restricted to the boundary, develop logarithmic singularities as $t \to \infty$, i.e.

$$|\partial_y W|_{y=0,1}| \gtrsim \log(t).$$

While such a point-wise estimate is sufficient to prove instability in C^0 and thus H^s for $s > 1/2$, it does not provide a description for y close to the boundary, which would, for example, be useful for the study of L^p spaces.

In the following, we therefore analyze the effect of the homogeneous correction on our solution and describe the asymptotic behavior close to the boundary. Here, for simplicity, we discuss only the evolution of $\partial_y W$, but all arguments can be adapted to study $\partial_y^2 W$ as well.

Recall that $\partial_y W$ evolves by (133):

$$\partial_t \partial_y W = \frac{if}{k} H^{(1)} + \frac{if}{k}\Phi^{(1)} + \frac{if'}{k}\Phi.$$

In view of the considerations on linearized Couette flow in Section 6 and as $\Phi^{(1)}$ and Φ vanish at the boundary and have a good structure, we in the following focus on the asymptotic behavior of

$$\frac{if(y)}{k}\int^T H^{(1)}(t,y)dt,$$

as $T \to \infty$ and for y close to the boundary.

LEMMA 5.13. *Let $T > 1$ and let u_1, u_2 be the solutions of*

$$(-1 + \left(g\frac{\partial_y}{k}\right)^2)u = 0,$$

with boundary values

$$\begin{aligned} u_1(0) = u_2(1) = 1, \\ u_1(1) = u_2(0) = 0. \end{aligned}$$

Then for any $y \in [0,1]$

$$\int_1^T H^{(1)}(t,y)dt = \int_1^T H^{(1)}(t,0)e^{ikty}dt\ u_1(y) + \int_1^T H^{(1)}(t,1)e^{ikt(y-1)}dt\ u_2(y).$$

PROOF. It has be shown in the previous sections that

$$H^{(1)}(t,y) = H^{(1)}(0,t)e^{ikty}u_1(y) + H^{(1)}(1,t)e^{ikt(y-1)}u_2(y).$$

Integrating in time then yields the result. □

In Section 2, we have shown that under H^s stability assumptions ,

$$H^{(1)}(0,t) = \frac{\omega_0}{g^2}\bigg|_{y=0}\frac{1}{t} + \mathcal{O}(t^{-1-s}),$$

and therefore, for $y = 0$,

$$\int_1^T H^{(1)}(t,0)e^{ikty}dt|_{y=0} = \frac{\omega_0}{g^2}|_{y=0}\int_1^T \frac{e^{ikty}}{t}dt|_{y=0} + \mathcal{O}(1) \gtrsim \log(T).$$

The case $y > 0$ is considered in the following lemma, where for convenience of notation we additionally assume that $k > 0$.

LEMMA 5.14. *Let $k > 0$, then for any $y \geq \frac{1}{2k}$,*

$$\int_1^T \frac{e^{ikty}}{t}dt$$

is bounded uniformly in T, k and y.

For any $0 < y < \frac{1}{2k}$,

$$\left|\int_1^T \frac{e^{ikty}}{t}dt\right| \lesssim \min(\log(T), -\log(ky)) + \mathcal{O}(1).$$

Further restricting to $0 < y < \frac{1}{2kT}$, also

$$\Re\left(\int_1^T \frac{e^{ikty}}{t}dt\right) \gtrsim \log(T) + \mathcal{O}(1).$$

Letting T tend to infinity, the logarithmic singularity persists:

$$\left|\int_1^\infty \frac{e^{ikty}}{t}dt\right| \gtrsim -\log(ky) + \mathcal{O}(1),$$

for $0 < y < \frac{1}{k}$.

PROOF. By a change of variables, $t \mapsto \tau = kyt$,

$$\int_1^T \frac{e^{ikty}}{t}dt = \int_{ky}^{kyT} \frac{e^{i\tau}}{\tau}d\tau.$$

Let thus $x_1, x_2 \geq 1/2$ be arbitrary but fixed, then

$$\int_{x_1}^{x_2} \frac{e^{i\tau}}{\tau}d\tau = \left.\frac{e^{i\tau}}{i\tau}\right|_{\tau=x_1}^{x_2} - \int_{x_1}^{x_2} \frac{e^{i\tau}}{i\tau^2}d\tau \lesssim \frac{1}{x_1} \leq 2.$$

Letting $x_1 = ky$ for $y \geq \frac{1}{2k}$, then proves the first result.

Let now $0 < ky < \frac{1}{2}$. In the case that $kyT > 1$, we can choose $x_1 = 1$ and $x_2 = kyT$ in the above estimate and thus obtain

$$\int_1^{kTy} \frac{e^{ikty}}{t}dt = \mathcal{O}(1).$$

It hence suffices to consider

$$\int_{ky}^{\min(kyT,1)} \frac{e^{i\tau}}{\tau}d\tau.$$

As $\tau \in (0,1)$,

$$0 < \cos(1) \leq \Re(e^{i\tau}) \leq 1$$

does not yield cancellations. Thus, the integral is comparable to

$$\begin{aligned}\int_{ky}^{\min(kyT,1)} \frac{1}{\tau}d\tau &= \log(\min(kyT,1)) - \log(ky)\\ &= \min(\log(kyT) - \log(ky), -\log(ky))\\ &= \min(\log(T), -\log(ky)).\end{aligned}$$

Letting T tend to infinity,

$$\lim_{T\to\infty} \min(\log(T), -\log(ky)) = -\log(ky),$$

which proves the last result. □

We have thus shown that, as $T \to \infty$, for y close to zero

$$\left|\int_1^T H^{(1)}dt\right| \gtrsim |\log(ky)| + \mathcal{O}(1).$$

In particular, while the L^∞ norm diverges, for any $1 \leq p < \infty$,

$$\log(y) \in L^p([0,1]),$$

and thus no blowup occurs in these spaces.

In view of our stability results for fractional Sobolev spaces, a natural question concerns the behavior of (fractional) y derivatives. Here we consider

$$C_s(T,y) := \int_1^T t^s \frac{e^{ikty}}{t}dt, \tag{134}$$

for $s \in (0,1)$ as a simplified interpolated model between

$$\int_1^T \frac{e^{ikty}}{t} dt,$$

and

$$\frac{d}{dy} \int_1^T \frac{e^{ikty}}{t} dt = ik \int_1^T e^{ikty} dt = \frac{e^{ikTy} - e^{iky}}{y}. \tag{135}$$

We note that, letting T tend to infinity in (135), the singularity is of the form

$$\frac{1}{y},$$

which is not in $L^p([0,1])$ for any $1 \le p \le \infty$. The intermediate cases $0 < s < 1$ are considered in the following lemma.

LEMMA 5.15. *Let $0 < s < 1$ and let $C_s(T,y)$ be given by* (134)*. Then*

$$C_s(T,0) = \frac{T^s - 1}{s},$$

and for $0 < y < \frac{1}{2k}$,

$$C_s(T,y) \lesssim \min(T^s, (ky)^{-s}) + \mathcal{O}(1).$$

For $0 < y < \frac{1}{2kT}$, also

$$\Re(C_s(T,y)) \gtrsim \frac{T^s - 1}{s} + \mathcal{O}(1).$$

Letting T tend to infinity, there exists a constant $c \in \mathbb{C}$, which is in general non-trivial, such that

$$C_s(\infty, y) = c(ky)^{-s} + \mathcal{O}(1).$$

PROOF. For $y = 0$, we compute

$$\int_1^T t^s \frac{1}{t} dt = \frac{t^s}{s}|_{s=1}^T = \frac{T^s - 1}{s}.$$

Controlling e^{ikty} by its absolute value, this also provides an upper bound for all $y > 0$.

Considering $y > 0$, we introduce a change of variables $t \mapsto kyt$

$$\int_1^T t^s \frac{e^{ikty}}{t} dt = (ky)^{-s} \int_{ky}^{kyT} \frac{e^{i\tau}}{\tau^{1-s}} d\tau, \tag{136}$$

which suggests a boundary singularity of the form $\min((ky)^{-s}, T^s)$. We first estimate

$$\int_{ky}^{kyT} \frac{e^{i\tau}}{\tau^{1-s}} d\tau$$

from above. In the case $x_1, x_2 \ge 1$, we integrate $e^{i\tau}$ by parts and thus obtain an estimate by

$$\left| \int_{x_1}^{x_2} \frac{e^{i\tau}}{\tau^{1-s}} d\tau \right| \lesssim \frac{1}{x_1^{1-s}} \le 1, \tag{137}$$

which is uniform in k, y and T. For $x_1, x_2 \le 1$ it suffices to estimate by the absolute value:

$$\left| \int_{x_1}^{x_2} \frac{1}{\tau^{1-s}} d\tau \right| \lesssim \frac{1}{s} x_2^s \le \frac{1}{s}. \tag{138}$$

Hence, by equation (136),

$$\left|\int_1^T t^s \frac{e^{ikty}}{t} dt\right| \lesssim (ky)^{-s}(1+\frac{1}{s}).$$

If ky is very small, i.e. $0 < y < \frac{1}{kT}$, then again $e^{i\tau}$ does not oscillate and the real part of the integral in (136) is comparable to

$$\int_{ky}^{kyT} \frac{1}{\tau^{1-s}} d\tau = \frac{1}{s}(T^s - 1)(ky)^s.$$

More precisely, we estimate

$$\cos(1) \leq \Re(e^{i\tau}) \leq 1.$$

We thus obtain a lower bound of

$$\Re(C_s(T,y))$$

by

$$\cos(1)(ky)^{-s}\frac{1}{s}(T^s-1)(ky)^s = \cos(1)\frac{1}{s}(T^s-1).$$

We again consider (136): Then by (137) the limit $T \to \infty$ exists as an improper integral. We thus have to show that

$$\int_{ky}^{\infty} \frac{e^{i\tau}}{\tau^{1-s}} d\tau = c + \mathcal{O}(|ky|^s)$$

for some $c \in \mathbb{C}$, which is in general non-trivial. By (138),

$$\lim_{y \downarrow 0} \int_0^{\infty} \frac{e^{i\tau}}{\tau^{1-s}} d\tau =: c,$$

exists.

Splitting and again using (138),

$$\int_{ky}^{\infty} \frac{e^{i\tau}}{\tau^{1-s}} d\tau = \int_0^{\infty} \frac{e^{i\tau}}{\tau^{1-s}} d\tau - \int_0^{ky} \frac{e^{i\tau}}{\tau^{1-s}} d\tau = c + \mathcal{O}(|ky|^s).$$

Thus, by equation (136),

$$C(\infty, y) = (ky)^{-s} \int_{ky}^{\infty} \frac{e^{i\tau}}{\tau^{1-s}} d\tau = c(ky)^{-s} + \mathcal{O}(1).$$

□

Letting T tend to infinity, we thus have to control a singularity of the form y^{-s}.

LEMMA 5.16. *Let $0 < s < 1$ and let $1 \leq p < \infty$, then*

$$y^{-s} \in L^p([0,1])$$

if and only if $p < \frac{1}{s}$.

The above result suggests that, for $1 \leq p < \infty$,

$$\sup_{T>1} \left\| \int_1^T H^{(1)} dt \right\|_{W^{s,p}}$$

is finite for $0 < s < \frac{1}{p}$ and infinite for $\frac{1}{p} < s < 1$. For the case $p = 2$, we have shown in Section 2, that indeed $s = \frac{1}{2}$ is critical in this sense.

CHAPTER 6

Consistency and nonlinear inviscid damping

In this final chapter, we briefly summarize our results and comment on the settings considered and the assumptions we imposed. Furthermore, we discuss consistency and implications for the nonlinear dynamics and the available results in the literature.

1. Spectral stability

As we remarked in Section 2.1, Rayleigh's stability criterion is better understood as a necessary but not sufficient criterion for spectral instability. It is however generally considered to be not very far from sufficient and we thus discuss the implications of our results for spectral stability of strictly monotone shear flows.

A result in this direction has previously been obtained by Hirota, Morrison, Hattori, [**HMH14**], who show that a shear flow U is spectrally stable, if it satisfies two assumptions:

- U is analytic and a bounded function on $[0, 1]$.
- U is strictly monotone and, if $U''(y_I) = 0$ at $y = y_I$, then $U'''(y_I) \neq 0$.

They further remark that these conditions can probably be relaxed.

As our L^2 stability result, Theorem 4.13, in particular implies spectral stability, we only need to require that

- $U'' \in W^{1,\infty}$ and U' is bounded.
- U is strictly monotone.
- A smallness condition on
$$\|U''(U^{-1}(\cdot))\|_{W^{1,\infty}} L$$
is satisfied, where L is the period in x.

Theorem 6.1 (Spectral stability). *Let U be strictly monotone and satisfy the assumptions of Theorem 4.13. Then, for $|k|$ sufficiently large, U is spectrally stable.*

Proof. Suppose to the contrary that there exists a non-trivial exponentially growing solution $\omega(t, y) = e^{\lambda t}\omega_0(y)$, $\omega_0 \in L^2$, $\Re(\lambda) > 0$. Then, by the L^2 stability result, Theorem 4.13,
$$e^{\Re(\lambda)t}\|\omega_0\|_{L^2} = \|\omega(t)\|_{L^2} \lesssim \|\omega_0\|_{L^2}.$$
Thus $e^{\Re(\lambda)t} \lesssim 1$ for all $t \in \mathbb{R}_+$, which contradicts $\Re(\lambda) > 0$. □

2. Periodic channels and separation in frequency

In this section, we briefly discuss the assumption of periodicity in the linearized problem and the relation to an infinitely long channel.

As we consider the behavior close to shear flows, the underlying domain has to be invariant under generic shears and thus has to be of the form
$$\mathbb{R} \times A$$

or

$$\mathbb{T} \times A$$

for some set $A \subset \mathbb{R}$.

In both settings, the linearized Euler equations around a shear flow $U(y)$ are given by

$$\begin{aligned}\partial_t \omega + U(y)\partial_x \omega &= U''(y)\partial_x \phi,\\ (\partial_x^2 + \partial_y^2)\phi &= \omega,\end{aligned}$$

where ϕ is required to be periodic in x for $x \in \mathbb{T}$ or square-integrable for $x \in \mathbb{R}$ and boundary conditions or integrability conditions are introduced in y, depending on the specific choice of A.

As the problem separates if A is not connected, we may restrict to three distinct cases:

- $A = \mathbb{R}$, the infinite channel (or whole space). This setting has the advantage of a vast existent literature for general kinetic transport equations, availability of many useful tools such as a Fourier transform and the lack of boundary.
- $A = [0, 1]$, the finite channel. Here several tools such as the Fourier transform are not available and boundary conditions play a non-negligible role as seen in Chapters 4 and 5. However A is now a compact set and in particular has finite measure, which is useful when working with Lebesgue spaces.
- $A = [0, \infty)$, the half-infinite channel, is not handled explicitly in this work, but can be either considered as a sub-case of the infinite channel, via a suitable extension, or as a limiting case for the finite channel.

Remark 12. *In principle, we could also consider the case of perturbations being periodic in y. However, the transport by*

$$\partial_t + U(y)\partial_x,$$

does not preserve this periodicity, unless $U(y)$ is also periodic with the same period as $\mathbb{T}$, which is not possible for a strictly monotone flow. It would thus only make sense to require that $W(t, x, y) = \omega(t, x - tU(y), y)$ is periodic. This can then be identified with a subcase of $A = [0, 1]$ and to our knowledge has no particular physical relevance.

Concerning the choice of either $x \in \mathbb{T}$ or $x \in \mathbb{R}$, we note that none of the coefficient functions, $U(y)$ and $U''(y)$, depend on x explicitly and are in particular both defined for all $x \in \mathbb{R}$ and periodic with respect to x.

Furthermore, fixing y, our equation decouples with respect to x in the sense that after a Fourier transform or basis expansion the frequency in x behaves as a parameter and there is no coupling between different frequencies $k_1 \neq k_2$:

$$\begin{aligned}\partial_t \hat{\omega} + U(y)ik\hat{\omega} &= U''(y)ik\hat{\phi},\\ (-k^2 + \partial_y^2)\hat{\phi} &= \hat{\omega},\\ (k, y) &\in \mathbb{Z} \times A \text{ or } \mathbb{R} \times A.\end{aligned}$$

On this level, the only difference between a periodic channel and an infinitely long channel is thus given by the fact that in the first case k may only take discrete values, while in the second $k \in \mathbb{R}$.

Considering the homogeneity in k of the right-hand-side and k close to but not equal to zero,

$$\hat{\omega} \mapsto k\hat{\phi}$$

has a large operator norm. For such perturbations, the underlying heuristic of our proof that the dynamics are close to transport is thus not valid anymore. For this reason we impose a frequency cut-off away from zero, i.e. $\langle \omega_0 \rangle(y)$ is incorporated into the shear flow and we require that

$$\omega - \langle \omega \rangle_x$$

is supported in frequencies k away from zero.

Under this assumption, all our results follow for the case of a non-periodic infinitely long channel, i.e. $x \in \mathbb{R}$, as well.

However, while some form of cut-off might be physically plausible also in the case of an (infinitely) long narrow channel, e.g. tides can probably be mostly neglected in a narrow channel, this cutoff will in general not be sharp but rather rapidly decaying. The assumed periodicity thus provides a more convenient and natural low-frequency cut-off mechanism.

3. Consistency and inviscid damping in a channel

A natural question, following the results on linear inviscid damping, concerns the behavior of the full nonlinear dynamics. We prove the following three results:

- Consistency: The linear dynamics are consistent, i.e. the nonlinearity, when evolved by the linear dynamics, is an integrable perturbation (in a less regular space). In the case of non-fractional Sobolev spaces and the infinite periodic channel, this has been already addressed in Section 4.
- Approximation: Supposing nonlinear inviscid damping holds in a space containing H^s, $s > 5$, we show that the solution remains in an H^{s-5} neighborhood of a linear solution (with $U(t,y)$ varying in time) uniformly in time.
- Instability: As a consequence, we show that, in a finite periodic channel, the stability result associated to nonlinear inviscid damping can generally not hold in high Sobolev spaces. Specifically we show that otherwise $\partial_y W$ would in general develop a logarithmic singularity at the boundary, which yields a contradiction.

The last result in particular implies that a Gevrey regularity result such as in [**BM13b**] would have to be heavily modified in the setting of a finite channel.

We first consider consistency, i.e. the evolution of the nonlinear term under the linear dynamics and shows that this would provide a uniformly controlled correction in Duhamel's formula.

Theorem 6.2 (Consistency). *Let W be a solution of the linearized Euler equations,* (122)*, in either the finite or infinite channel, with*

$$\int \omega_0(x,y)dx \equiv 0,$$

$f, g \in W^{3,\infty}$ *and assume that for some $s \in (2,3)$*

$$\|W(t)\|_{H^s} < C < \infty,$$

is uniformly bounded (e.g. via Theorem 4.11 or Theorem 5.4). Then,

$$\|v \cdot \nabla \omega\|_{L^2} = \mathcal{O}(t^{-(s-1)}).$$

In particular,

$$W(t) + \int^t \nabla^{\perp}\Phi(\tau)\nabla W(\tau)d\tau$$

remains in a bounded neighborhood of $W(t)$ and there exist asymptotic profiles $W_{\pm\infty,con} \in L^2$ such that

$$W(t) + \int^t \nabla^\perp \Phi(\tau) \nabla W(\tau) d\tau \xrightarrow{L^2} W_{\pm\infty,con},$$

as $t \to \pm\infty$.

Proof. Following the same approach as in Section 4

$$\|v \cdot \nabla \omega\|_{L^2} = \|\nabla^\perp \Phi \nabla W\|_{L^2}.$$

As $s > 2$ (and we consider two spatial dimensions, x, y), we can use a Sobolev embedding to control

$$\|\nabla W\|_{L^\infty_{xy}(\Omega)} \lesssim \|W\|_{H^s}.$$

It thus suffices to estimate

$$\|\nabla^\perp \Phi\|_{L^2}.$$

Taking the $\nabla^\perp$ into account and using (an interpolated version of) the damping results of Section 1,

$$\|\nabla^\perp \Phi\|_{L^2} = \mathcal{O}(t^{-(s-1)}) \|W\|_{H^s}.$$

As $s - 1 > 1$, this decay is integrable, which together with the scattering result for $W(t)$ of Chapter 4 concludes the proof. □

We remark that this consistency result loses regularity and indeed controlling the loss of regularity due to the nonlinearity is one of the main challenges in the nonlinear problem, as we briefly discuss in Section 4.

While the linear dynamics are thus consistent in the above sense, higher regularity and how well they approximate the nonlinear dynamics is not answered by the preceding theorem.

In the following, we consider the converse problem, i.e. given a nonlinearly stable solution with inviscid damping, we estimate the effect of the nonlinearity. For this purpose, we note that the 2D Euler equations

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= 0, \\ v &= \nabla^\perp \phi, \\ \Delta \phi &= \omega, \end{aligned}$$

on either the infinite or finite periodic channel possess a good structure with respect to x averages. Denote

$$\begin{aligned} \omega &= (\omega - \langle \omega \rangle_x) + \langle \omega \rangle_x = \omega' + \langle \omega \rangle_x, \\ \phi &= (\phi - \langle \phi \rangle_x) + \langle \phi \rangle_x = \phi' + \langle \phi \rangle_x. \end{aligned}$$

Then,

$$\begin{aligned} \partial_t \omega' - (\partial_y \langle \phi \rangle_x) \partial_x \omega' + (\partial_y \langle \omega \rangle_x) \partial_x \phi' &= (\nabla^\perp \phi' \cdot \nabla \omega')', \\ \partial_t \langle \omega \rangle_x &= \langle \nabla^\perp \phi' \cdot \nabla \omega' \rangle_x. \end{aligned}$$

In analogy to the linear setting, we denote

$$-\partial_y \langle \phi \rangle_x =: U(t, y),$$

and for the moment restrict our attention to the first equation, considering $U(t, y)$ as given.

In this formulation the Euler equations then read

$$\partial_t \omega' + U(t, y) \partial_y \omega' = (\partial_y^2 U(t, y)) \partial_x \phi' + (\nabla^\perp \phi' \cdot \nabla \omega')'.$$

Further introducing the volume-preserving change of variables

$$(x,y) \mapsto (x - \int_0^t U(\tau,y)d\tau, y)$$

and defining W, Φ via these coordinates, the *Euler equations in scattering formulation* are given by

$$\partial_t W = (\partial_y^2 U(t,y))\partial_x \Phi + \nabla^\perp \Phi \cdot \nabla W. \tag{ES}$$

Obtaining a good control of the regularity of $U(t,y)$ as well as appropriate decay is a very hard problem, in particular as the evolution of $U(t,y)$ and W is coupled. In the following theorem, such control is therefore *assumed.*

THEOREM 6.3 (Approximation). *Let $W(t,x,y)$ be a solution of* (ES) *and suppose that, for some $s > 2$, inviscid damping holds in H^s with integrable rates, i.e. suppose that, for some $\epsilon > 0$,*

$$\|\nabla^\perp \Phi\|_{H^s} = \mathcal{O}(t^{-1-\epsilon})\|W\|_{H^{s+2+\epsilon}}.$$

Suppose further that $\|W(t)\|_{H^{s+2+\epsilon}}$ is uniformly bounded. Then,

$$\|\nabla^\perp \Phi \cdot \nabla W\|_{H^s} = \mathcal{O}(t^{-1-\epsilon}),$$

and, in particular,

$$\int_0^t \|\nabla^\perp \Phi(\tau) \cdot \nabla W(\tau)\|_{H^s} d\tau$$

is bounded uniformly in t and converges as $t \to \infty$.

PROOF. As $s > 2$, H^s forms an algebra and

$$\|\nabla^\perp \Phi \cdot \nabla W\|_{H^s} \leq \|\nabla^\perp \Phi\|_{H^s}\|\nabla W\|_{H^s} = \mathcal{O}(t^{-1-\epsilon}),$$

which proves the result. □

REMARK 13.
- *The results of Section 1.1 can be extending to provide sufficient conditions for inviscid damping with integrable rates to hold. The core problem of inviscid damping is thus again the control of the regularity of W.*
- *If $\|W\|_{H^{s+2+\epsilon}} < \delta$ is small, then*
  $$\int_0^t \|\nabla^\perp \Phi(\tau) \cdot \nabla W(\tau)\|_{H^s} d\tau = \mathcal{O}(\delta^2)$$
  is quadratically small. The linearization thus remains valid, but only in a less regular space. For this reason we call this theorem an "approximation" result.
- *Even if $\|W\|_{H^{s+2+\epsilon}}$ is not small, the nonlinearity yields a bounded contribution. Hence, if*
  $$\| \int_0^t (\partial_y^2 U(\tau,y))\partial_x \Phi(\tau) d\tau\|_{H^s}$$
  grows unboundedly as $t \to \infty$, i.e. the linear part is unstable, then, as shown in the following theorem, the nonlinear dynamics can not be stable.

THEOREM 6.4 (Instability). *Let W be a solution of* (ES), $\partial_y^2 U(t,y) \in W^{1,\infty}_{y,t}$ *and suppose that*

$$\partial_y^2 U(t,y)|_{y=0} > c > 0$$

for all $t > 0$. Suppose further that for some k

$$|\mathcal{F}_x(\partial_y(\nabla^\perp \Phi \cdot \nabla W))(t,k,0)| = \mathcal{O}(t^{-1-\epsilon}),$$

and

$$\mathcal{F}_x W(t,k,y)|_{y=0} > c > 0,$$

for all time.

Then,

$$|(\mathcal{F}_x \partial_y W)(t,k,0)| \gtrsim \log(t),$$

as $t \to \infty$.

In particular, for any $s > 2$, $\|W\|_{H^s_{x,y}}$ then can not be bounded uniformly in time.

Proof. Differentiating (ES) with respect to y, we obtain that $\partial_y W$ satisfies

$$\partial_t \partial_y W = \partial_y \left(\partial_y^2 U(t,y) \partial_x \Phi\right) + \partial_y (\nabla^\perp \Phi \cdot \nabla W).$$

Restricting to $y = 0$ and using that $\partial_x \Phi$ vanishes on the boundary, as it is assumed to be impermeable, we consider the k Fourier mode. Then,

$$\partial_t \mathcal{F}_x \partial_y W(t,k,0) = \partial_y^2 U(t,0) ik (\mathcal{F}_x \partial_y \Phi)(t,k,0) + \mathcal{O}(t^{-1-\epsilon}). \tag{139}$$

Similar to the previous chapters, $\mathcal{F}_x \Phi$ solves a shifted elliptic equation:

$$\left(-k^2 + \left(\partial_y - itk \int^t \partial_y U(\tau,y) d\tau\right)^2\right) \mathcal{F}_x \Phi = \mathcal{F}_x W.$$

A homogeneous solution u of this equation is then of the form

$$u(t,y) = \exp\left(\int^t (U(\tau,y) - U(\tau,0)) d\tau\right) u(0,y).$$

By the same argument as in Lemma 5.6, $\mathcal{F}_x \Phi(t,0)$ can hence be computed in terms of

$$\langle \mathcal{F}_x W, u(t,y) \rangle_{L^2}, \tag{140}$$

where we assumed that

$$u(0,0) = 1, u(0,1) = 0.$$

Integrating

$$u(t,y) = u(0,y) \frac{1}{\int^t \partial_y U(\tau,y) d\tau} \partial_y \exp\left(\int^t (U(\tau,y) - U(\tau,0)) d\tau\right)$$

by parts in (140), then yields a leading order term of the form

$$\left| \frac{1}{\int^t \partial_y U(\tau,y) d\tau} W|_{y=0} \right| \gtrsim \frac{c}{t}.$$

Integrating (139) in time thus yields a logarithmic singularity and hence the result. □

We remark that, using a Sobolev embedding, the decay of

$$|\mathcal{F}_x(\partial_y(\nabla^\perp \Phi \cdot \nabla W))(t,k,0)|$$

is a consequence of inviscid damping in a high Sobolev space. Furthermore, restricting (ES) to the boundary,

$$\mathcal{F}_x W(T,k,0) = \mathcal{F}_x \omega_0(t,k,0) + \int_0^T \mathcal{F}_x \left(\nabla^\perp \Phi \cdot \nabla W\right)(t,k,0) dt.$$

If one thus assumes $(\mathcal{F}_x \nabla^\perp \Phi \cdot \nabla W)(t,k,0)$ to decay with an integrable rate, then, for $\mathcal{F}_x \omega_0(t,k,0)$ sufficiently large, also

$$\mathcal{F}_x W(T,k,0)$$

will be bounded away from zero uniformly in T.

The theorem thus implies that, in the generic case, solutions of (ES) can not remain bounded in any high Sobolev space, which embeds into $W^{1,\infty}$.

4. Nonlinear inviscid damping for Couette flow

In this section, we briefly comment on the additional challenges arising in the nonlinear setting and the work of Bedrossian and Masmoudi, [**BM13b**], on nonlinear inviscid damping for small Gevrey perturbations to Couette flow in an infinite periodic channel. Their main result is given by the following theorem:

THEOREM 6.5 ([**BM13b**, page 5]). *For all* $1/2 < s \leq 1$, $\lambda_0 > \lambda' > 0$ *there exists* $\epsilon_0 \leq 1/2$ *such that for all* $\epsilon \leq \epsilon_0$ *and all* ω_0 *such that*

$$\iint \omega_0 dxdy = 0,$$

$$\iint |y\omega_0(x,y)|dxdy < \epsilon,$$

$$\|\omega_0\|^2_{\mathcal{G}^{\lambda_0}} := \sum_k \int |\tilde{\omega_0}(k,\eta)|^2 e^{2\lambda_0|(k,\eta)|^2} d\eta \leq \epsilon^2,$$

then there exists f_∞ *with*

$$\iint f_\infty dxdy = 0,$$

$$\|f_\infty\|_{\mathcal{G}^{\lambda'}} \lesssim \epsilon,$$

such hat

$$\|\omega(t, x+ty+\Phi(t,y), y) - f_\infty(x,y)\|_{\mathcal{G}^{\lambda'}} \lesssim \frac{\epsilon^2}{< t >},$$

where

$$\Phi(t,y) = \int_0^t \langle v_1\rangle_x(\tau, y)d\tau = u_\infty(y)t + \mathcal{O}(\epsilon),$$

and

$$u_\infty = \partial_y^{-1}\langle f_\infty\rangle_x.$$

Moreover

$$\|\langle v_1\rangle_x - u_\infty\|_{\mathcal{G}^{\lambda'}} \lesssim \frac{\epsilon^2}{< t >^2},$$

$$\|v_1 - \langle v_1\rangle_x\|_{L^2} \lesssim \frac{\epsilon}{< t >},$$

$$\|v_2(t)\|_{L^2} \lesssim \frac{\epsilon}{< t >^2}.$$

As the proof is technically very challenging, we will not attempt a sketch but rather discuss some of the techniques, tools and challenges. The interested reader is referred to the expository article by Bedrossian and Masmoudi, [**BM13a**], for an overview of the proof and further discussion.

As in the linear setting, it is natural to consider coordinates moving with the underlying shear flow

$$(y + \langle v_1\rangle_x(t,y), 0),$$

which now depends on time.

The corresponding change of coordinates is thus given by

$$\begin{aligned}(t,x,y) &\mapsto (t,z,\nu),\\ z(t,x,y) &= x - t\nu,\\ \nu(t,y) &= y + \frac{1}{t}\int_0^t \langle v_1\rangle_x(\tau,y)d\tau,\end{aligned}$$

where the coordinate ν is chosen in such a way as to preserve the form $x - t\nu$.

We remark that this change of variables is not volume-preserving due to the change from y to ν and that its regularity is not anymore given explicitly by the initial shear flow $(y,0)$ but rather in terms of the regularity of $\nu(t,y)$.

In particular, as this change of variables influences the coefficients of the (shifted) elliptic equation of the stream function, elliptic regularity results depend on the regularity of the transformation. Therefore, the regularity of the change of variables has to be controlled in addition to the regularity of vorticity and velocity perturbation.

A more serious difficulty of the nonlinear problem is due to the loss of decoupling with respect to frequencies k in the periodic direction. More precisely, while the underlying shear still is independent of x and the linear terms thus decouple, the nonlinearity

$$\nabla^{\perp}\phi\cdot\nabla\omega$$

introduces a coupling between different frequencies.

We recall from Section 3 that the Orr mechanism and its associated growth include critical times, i.e. that in a simplified model case and moving with the flow

$$\tilde{\Phi}(t,k,\eta) = \frac{1}{k^2+(\eta-kt)^2}\tilde{W}(t,k,\eta),$$

and the multiplier is largest at the critical time $t = \frac{\eta}{k}$.

As the linear setting decouples, in that case one may restrict to k arbitrary but fixed and thus for every η there is only one critical time, which we show to yield a bounded contribution.

In this nonlinear case, however, any mode k could at its critical time have a large effect on all other frequencies, which then at their critical time in turn could have a large effect. Controlling this possible *cascade* is one of the main challenges of the nonlinear problem.

Bedrossian and Masmoudi introduce a toy model of the form

$$\begin{aligned}\partial_t f &= \partial_y(\phi - \langle\phi\rangle_x)\partial_x f_{lo},\\ (\partial_x^2 + (\partial_y - t\partial_x)^2)\phi &= f,\end{aligned}$$

where f_{lo} is a given function corresponding to the low frequency part of f.

Taking a Fourier transform in both x and y and, for simplicity, restricting to $\partial_x f_{lo}$ depending only on t and x, one obtains

$$\partial_t\tilde{f}(t,k,\eta) = \sum_{l\neq 0}\frac{\eta(k-l)}{l^2+|\eta-lt|^2}\tilde{f}(t,l,\eta)\tilde{f}_{lo}(t,k-l).$$

Consider now frequencies l close to k at their respective critical times t_l. The contribution due to that mode is roughly given by

$$\int_{t_l-\delta}^{t_l}\frac{\eta}{l^2+(\eta-lt)^2}dt\tilde{f}(t,l,\eta) \approx \frac{\eta}{l^2}\tilde{f}(t,l,\eta) \approx \frac{\eta}{k^2}\tilde{f}(t,l,\eta).$$

Starting at a critical time t_k and considering neighboring modes, each mode could then contribute growth by a factor $\frac{\eta}{k^2}$. Optimizing the size of the neighborhood and estimating from above by this worst case growth with all modes, Bedrossian and Masmoudi obtain an upper growth bound by $\mathcal{O}(e^{C\sqrt{\mu}})$.

The cascade is thus estimated, in the worst case, to at most amount to loss of Gevrey 2 regularity, which results in the requirement $s > 1/2$ in the theorem.

There are experimental indications that some form of cascade may happen. In [**YOD05**], *echoes* are observed, i.e. two perturbations of different frequencies strongly influence a third mode, which results in an additional peak after a given amount of time.

It is however not clear whether a full cascade (of a large number of modes) in the above sense is possible and whether better than Gevrey 2 regularity is necessary. In particular, as shown in Section 3, this would imply that, in a finite channel, nonlinear inviscid damping either does not hold or is very different from the infinite-channel case.

Indeed, the only known lower bound on the necessary regularity requirements is given by the work of Lin and Zeng, [**LZ11**]. There, it is shown that for Sobolev spaces H^s with $s < 3/2$, there exist non-trivial stationary solutions to the nonlinear problem in the form of "cat eyes" in every neighborhood. Therefore, asymptotic convergence to shear flow solutions can not hold unless one rules out these solutions, which is shown to be the case in small enough H^s neighborhoods for every $s > 3/2$. We stress that this is a nonlinear effect and that linear inviscid damping has been proven in Chapter 4 without such restrictions.

Summary

In this thesis, we study the linear stability and long-time asymptotic behavior of solutions to the 2D incompressible Euler equations

$$\begin{aligned} \partial_t \omega + v \cdot \nabla \omega &= 0, \\ \nabla \times v &= \omega, \\ \nabla \cdot v &= 0, \end{aligned}$$

which model the evolution of an inviscid, incompressible fluid.

The Euler equations possess many conserved quantities, among them the kinetic energy, the enstrophy and entropy, and in particular exhibit neither dissipation nor entropy increase. As shown by Arnold, [**Arn66b**], they even possess the structure of an infinite-dimensional Hamiltonian system on the Lie algebra of smooth volume-preserving diffeomorphisms. It was thus a very surprising observation of Kelvin, [**Kel87**], and later Orr, [**Orr07**], that small perturbations to Couette flow, i.e. the linear shear $v(t,x,y) = (y,0)$, are damped back to a (possibly different) shear flow. As the linearized Euler equations around Couette flow can be *solved explicitly*, direct computations show that under the linear dynamics

$$\begin{aligned} \|v_1(t) - \langle v_1 \rangle_x\|_{L^2} &\leq \mathcal{O}(t^{-1}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H_x^{-1} H_y^1}, \\ \|v_2(t)\|_{L^2} &\leq \mathcal{O}(t^{-2}) \|\omega_0 - \langle \omega_0 \rangle_x\|_{H_x^{-1} H_y^2}, \end{aligned}$$

and that these decay rates are sharp. This phenomenon is commonly called *inviscid damping* in analogy to *Landau damping* in plasma physics.

Going beyond these classic results, which due to the explicit solution are in a sense trivial, has, however, remained mostly open until recently. As the main result of this thesis, we prove that linear inviscid damping holds for a large class of regular monotone shear flows. There, we consider both the common setting of an infinite periodic channel, $\mathbb{T} \times \mathbb{R}$, as well as the physically relevant finite periodic channel, $\mathbb{T} \times [0,1]$, with impermeable walls. In the latter setting, boundary effects are shown to qualitatively change the behavior of solutions and that, in general, asymptotically (logarithmic) singularities develop on the boundary. In particular, regularity results with respect to y are thus limited to the critical fractional Sobolev spaces $H_y^{3/2}([0,1])$ for general perturbations and $H_y^{5/2}([0,1])$ for perturbations with vanishing Dirichlet data, $\omega_0|_{y=0,1} = 0$.

We further discuss the implications of our stability results for the problem of nonlinear damping, where high regularity would be essential to control nonlinear effects. In particular, we show that the stability results for the finite periodic channel and the associated instability in supercritical Sobolev norms provide an upper bound on the Sobolev regularity that can be controlled in the nonlinear setting. It is hence unclear whether the recent results of Bedrossian and Masmoudi, [**BM13b**], on nonlinear inviscid damping for Couette flow in an infinite periodic channel and under small Gevrey perturbations can be adapted to the setting of a finite periodic channel.

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